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Non-Uniform Hypercoherences

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Abstract

In [BE01], Bucciarelli and Ehrhard propose a general tool for building a wide class of models of linear logic where a formula is interpreted as a set (the web) together with a kind of phase valued "coherence relation". These interpretations are non-uniform in the sense that the semantics of a proof makes no assumption about the behaviour of its possible counter-proofs, unlike e.g. in the usual stable semantics where the argument of a stable functional is always a stable function. However, until now, it was suspected that this non-uniformity necessarily induces a kind of non-determinism, namely that a "clique" and an "anti-clique" could have more than one point in common. We provide a new non-uniform semantics of linear logic where this property of determinism is preserved. This is done by constructing the co-free exponential in the "non-uniform coherence space" framework described at the end of [BE01]. We discuss the issue of sequentiality in this new model.

Notations.

We use the notation $[\ ]$ for multisets while the notation $\{\ \}$ is, as usual, for sets. The pairwise union of multisets is denoted by a $+$ sign and following this notation the generalised union is denoted by a $\sum$ sign. The neutral element for this operation, the empty multiset, is denoted by $[]$. If $k \in \mathbb{N}$, $k[a]$ denotes the multiset $\sum_1^k [a]$. If $[a_i \mid i \in I]$ is a multiset, its support is the set $\{a_i \mid i \in I\}$. The cardinality $\sharp[a_i \mid i \in I]$ of a multiset $[a_i \mid i \in I]$ is the cardinality $\sharp I$ of the set I. If m is a multiset we denote by $\mathrm{supp}(m)$ its support. The disjoint sum operation on sets is defined by setting $A + B = \{1\} \times A \cup \{0\} \times B$. The categorical composition is denoted by $\fatsemi$.

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1 Introduction

Strong stability has been introduced by Bucciarelli and Ehrhard in [BE94] for the purpose of giving a purely "extensional" definition of sequentiality at all types, that is, a description of sequential computations which does not involve the atomic description of each step of interaction of an agent (function, term) with its environment (argument, or more generally, context), as game models do. The results obtained by Ehrhard in [Ehr99] (and later proved again by Longley and Van Oosten, with different methods, see [Lon02,vO97]) showed that indeed, strong stability corresponds to sequentiality at all types; Ehrhard established that the strongly stable model is the extensional collapse of the sequential algorithm model designed in the late 70's by Berry and Curien ([BC82]). Unlike the continuous or stable interpretations of PCF, the sequential algorithm interpretation (which is now better understood as a deterministic game model) is very "operational" in nature: Cartwright, Curien and Felleisen showed in [CCF94] that sequential algorithms are fully abstract (and fully complete) for the extension of PCF by a *catch and throw* mechanism. The intuition that strong stability is relevant from an operational viewpoint is further supported by recent results showing for instance that the strongly stable model is the extensional collapse of an extension of PCF with states ([Lai01]). In [Lon02], Longley advocates the claim that there exists a canonical notion of "sequential" functionals of all types which coincides with the hierarchy of strongly stable functions (in the "effective" hierarchy, the situation is more complicated however).

This comparison of the strongly stable model with more operational interpretations has been made possible only by the discovery of *hypercoherences* by Ehrhard ([Ehr93]). Moreover, the introduction of these objects simplified the presentation of the strongly stable semantics and provided a strongly stable interpretation of (second order) Linear Logic. A hypercoherence is very similar to a coherence space (see [Gir87]) and consists of a set (the web) together with a coherence relation on this web. However, in a hypercoherence, the coherence relation is not a binary relation, but a set of finite subsets of the web containing all singletons (these sets are said to be coherent). An "element" of a hypercoherence X is then a *clique* of X, that is, a subset of the web of X which has the property that all its finite and non-empty subsets are coherent.

Hypercoherences are a model of linear logic, so they provide an interpretation of intuitionistic implication which is of the shape $X \Rightarrow Y = (!X) \multimap Y$ where "$\multimap$" is a linear implication and ! is a so called "exponential". The basic operational intuition behind this decomposition is as follows: a linear map represents a program which uses its argument exactly once, and an element of $!X$ is obtained essentially by taking an element of X and making it available as many times as required.

The hypercoherences model is said to be static as opposed to games models which involve a direct representation of the dynamics of computation. In game semantics, time is explicit: such semantics interpret terms by focusing on the historic of an atomic interaction between a player (the program implemented by the term) and an opponent (the environment). For instance an interaction inside a function type $A \to B$ is an interleaving of an interaction querying a piece of A data and an interaction producing a piece of B data.

There is no such reference to time in the hypercoherences semantics; indeed, the web constructions in the hypercoherences interpretation follow the patterns of the purely relational model of linear logic. In particular the web of a linear function space is the Cartesian product of the webs.

However the strong relation of hypercoherences with sequentiality means that the model carries an *implicit* representation of time which is clearly missing in the purely relational model. So this implicit representation of time is to be found in the coherence relation.

Due to the mathematical simplicity of static models, we believe that a deeper understanding of this kind of implicit representation of time is of great interest when studying mathematical properties of computations.

But, when we tried to compare this implicit representation of time with games models were the time is explicit, a crucial point to deal with raises. This is known as the *uniformity of the exponentials* issue. The relational model almost consists of the part of the hypercoherence model dealing with webs, except that in hypercoherences the web of the exponentials depends on the coherence relations. The dependence of webs on coherence is what is called uniformity of the exponentials. The terminology comes from the fact that in such models the context of an agent behaves uniformly that is as if this context is produced by a single agent. Making the difference between the static information which concerns types and which is delivered by the web and the dynamic information which concerns terms and which is delivered by the coherence relation is hard since uniformity mixes them up. Moreover, due to uniformity the hypercoherence interpretation of a term misses points relatively to its relational interpretation and so the hypercoherence semantics loses information about some branches of the computation (the same holds for the coherence semantics). For instance the relational semantics of the simply typed term

$$\lambda b^{\mathbf{bool}}.\,\mathtt{if}\; b \;\mathtt{then}\; (\mathtt{if}\; b \;\mathtt{then}\; \mathtt{t} \;\mathtt{else}\; \mathtt{t}) \;\mathtt{else}\; (\mathtt{if}\; b \;\mathtt{then}\; \mathtt{f} \;\mathtt{else}\; \mathtt{f}).$$

(where $\mathtt{t}$ stands for true and $\mathtt{f}$ for false) is the relation

$$\{([\mathtt{t},\mathtt{t}],\mathtt{t}),([\mathtt{f},\mathtt{t}],\mathtt{t},([\mathtt{f},\mathtt{t}],\mathtt{f})([\mathtt{f},\mathtt{f}],\mathtt{f})\}$$

but its hypercoherences semantics is just $\{([\mathtt{t},\mathtt{t}],\mathtt{t}),([\mathtt{f},\mathtt{f}],\mathtt{f})\}$.

The hypercoherence semantics of the term trusts its environment and makes the assumption that the boolean b has one fixed value during the time of the computation. Of course, this is fair from an interactive viewpoint since the environment complies with coherence conditions as programs do. But for the purpose of reconstructing terms from their semantics, which is related with our goal, some information is missing.

Non-uniform static models will interpret terms exactly as the relational model does. This will allow us to *combine* models in order to take advantage of their different features.

The uniformity/non-uniformity issue in static semantics is to be related with games where some uniformity condition was originally designed for the exponential type (interactions in $!A$ are deterministic interleaving of interactions of A, see [AJM94]). Recent works in the game semantics area are more permissive: they postponed such conditions on the semantics of types to conditions on the semantics of terms.

Providing coherence or hypercoherence semantics with non-uniform exponentials is not a trivial job. One has to design a semantics where for instance, one point of the web shall be incoherent with himself. This must be the case for the point $[\mathtt{t}, \mathtt{f}]$ since the valid term above maps it to an incoherent piece of data $\{\mathtt{t}, \mathtt{f}\}$. The situation where two different points are coherent and incoherent at the same time may also arise (this will mean the semantics does not enjoy *determinism*, we come back with this latter).

A. Bucciarelli and T. Ehrhard have designed a general tool for producing non-uniform semantics (see [BE01]). As observed by J.-Y. Girard in [Gir96], to be closer to full completeness for linear logic, the coherence spaces semantics can be enriched by indexing each clique on a monoid. To make the story short, by doing this and thanks to a clever handling of *indexes* (locations), A. Bucciarelli and T. Ehrhard obtained that when this monoid comes with a *phase space* structure of a certain sort (a symmetric phase space which is a truth-value model of an *indexed* linear logic calculus, in fact) this leads to a denotational model of linear logic. For details see [BE00] and [BE01]. This leaves us with, potentially, an infinity of denotational models of linear logic. A. Bruasse-Bac has studied many of them in her ph.D. thesis ([Bru01]) among which there is one rejecting the *Mix* rule. A quite simple phase space produces non-uniformity for coherence semantics. By generalizing this construction to all arities one obtains non-uniformity for something like hypercoherence semantics. But this latter semantics badly relates with usual hypercoherences and the model misses some important features (*e.g.* one can define "Gustave functions").

We have investigated these latter semantics. We discovered that many variants of their exponentials are possible. Among them we found the *co-free* ! exponential (think of it as to be an *infinite* tensor product) which is the maximal exponential in a sense we shall make precise in corollary 1. Even with this maximal solution, the non-uniform semantics admits a non sequentially definable morphism.

Our non-uniform models enjoy the property that the intersection of a clique and an anti-clique contains at most one point, like in the uniform semantics. This means that the relational model is *deterministic* in the following sense.

Consider the linear logic system expanded with para-rules such that the system still enjoy cut-elimination and such that the semantics extends to this new system. In this system a formula A and its linear negation $A^\perp$ can be both provable. Then the relational interpretations of a proof of A and of a proof of $A^\perp$ can interact on at most one point through cut-elimination.

We derive a uniform model from each of our non-uniform models just by restriction to the part of the web where these interactions take place. As a consequence, we can extract *in one step* the uniform interpretation of a type or a term from its non-uniform interpretation. The uniform semantics defined using non-uniform coherence spaces is the usual coherence model. The general uniform semantics is a new one, *multicoherences*. Hypercoherences are particular multicoherences and we use this to show how to define non-uniform hypercoherences enjoying the determinism and uniformity "in one step" properties.

The multicoherence model has still to be explored. At a first income we have that: at first order simple types, each finite clique of this model is subdefinable; each clique of the multicoherence model is a clique of the coherence space model; and, at a functional type, there exists sets which are cliques in the hypercoherence model but which are not cliques in the multicoherence model. So there is at least two extensionally different notions of higher order sequentiality.

2 K-coherence semantics of Linear Logic

2.1 The relational semantics

We recall briefly the interpretation of linear logic in the category of sets and relations.

Formulae. A formula A is interpreted by a set $|A|$ defined inductively as follows: $|0| = |\top| = \emptyset$, $|1| = |\bot| = \{*\}$, $|A^\perp| = |A|$, $|A \oplus B| = |A \& B| = |A| \cup |B|$ if $|A| \cap |B| \neq \emptyset$ or $|A| + |B|$ otherwise, $|A \otimes B| = |A \parr B| = |A| \times |B|$ and $|!A| = |?A| = \mathcal{M}_{\mathrm{fin}}(|A|)$ where $\mathcal{M}_{\mathrm{fin}}(E)$ is the set of finite multisets on E.

Sequents. We use the right-sided presentation of the linear logic sequent calculus. Up to associativity and commutativity of the Cartesian product, the "comma" of sequents is safely interpreted as a *par i.e.* by setting $|\vdash A_1, \ldots, A_n| = |A_1 \parr \ldots \parr A_n|$ which is equal to $|A_1| \times \ldots \times |A_n|$.

Proofs. The interpretation of a proof of a sequent $\vdash \Gamma$ is a subset of $|\vdash \Gamma|$ defined inductively on the proof, by case on the last rule, as show below.

It is well-known that this interpretation is a denotational semantics of linear logic (that is: two proofs of a given sequent have the same interpretation as soon as they are equivalent up to cut-elimination).

identity group

$$\frac{}{\vdash A, A^\perp : \{(a,a) \mid a \in |A|\}} \quad \frac{\vdash \Gamma, A : f \qquad \vdash \Delta, A^\perp : g}{\vdash \Gamma, \Delta : \{(\gamma,\delta) \mid \exists a, (\gamma,a) \in f \wedge (\delta,a) \in g\}}$$

additives

$$\frac{}{\vdash \Gamma, \top : \emptyset} \qquad \frac{\vdash \Gamma, A : f \qquad \vdash \Gamma, B : g}{\vdash \Gamma, A\&B : f \cup g} \qquad \frac{\vdash \Gamma, A : f}{\vdash \Gamma, A \oplus B : f}$$

multiplicatives

$$\frac{\vdash \Gamma : f}{\vdash \Gamma, \perp : f \times \{*_\perp\}} \qquad \frac{}{\vdash 1 : \{*_1\}}$$

$$\frac{\vdash \Gamma, A, B : f}{\vdash \Gamma, A \parr B : f} \qquad \frac{\vdash \Gamma, A : f \qquad \vdash \Delta, B : g}{\vdash \Gamma, \Delta, A \otimes B : \{(\gamma, \delta, (a,b)) \mid (\gamma,a) \in f, (\delta,b) \in g\}}$$

exponentials

$$\frac{\vdash ?A_1, \ldots, ?A_n, A : f}{\vdash ?A_1, \ldots, ?A_n, !A : f^\dagger} \qquad \frac{\vdash \Gamma, ?A, ?A : f}{\vdash \Gamma, ?A : \{(\gamma, x_1 + x_2) \mid (\gamma, x_1, x_2) \in f\}}$$

$$\frac{\vdash \Gamma : f}{\vdash \Gamma, ?A : \{(\gamma, []) \mid (\gamma) \in f\}} \qquad \frac{\vdash \Gamma, A : f}{\vdash \Gamma, ?A : \{(\gamma, [a]) \mid (\gamma, a) \in f\}}$$

Where $f^\dagger$ is equal to :

$$\{(\sum_{1 \le j \le k} x_1^j, \ldots, \sum_{1 \le j \le k} x_n^j, [a_1, \ldots, a_k]) \mid k \in \mathbb{N}, \forall j, (x_1^j, \ldots, x_n^j, a_j) \in f\}.$$

2.2 K-coherence spaces

We introduce a general notion which will provide us with a very convenient language for describing the various models we deal with. A *power* is simply a functor from the category of sets and inclusions to itself. Typical powers relevant to our purpose are:

- the empty power P defined by $P(E) = \emptyset$. This power will simply be denoted $\emptyset$;

- the non-empty finite sets power $\mathcal{P}^*_{\mathrm{fin}}$ which maps each set to the set of its finite non-empty subsets (it will be used for dealing with hypercoherences);
- given a subset K of $\mathbb{N} \setminus \{0,1\}$, the power $\mathcal{M}_K$ which maps a set E to the set of all finite multisets over E whose cardinality belongs to K ($\mathcal{M}_{\{2\}}$ will be used for dealing with coherence spaces). The choice of this power follows the suggestion made at the end of [BE01] for the purpose of building non-uniform coherence or hypercoherence like models.

Definition 2.1 Let P be a power. A *P-coherence space* X is given by a triple $(|X|, \overset{\frown}{\smile}_X, \asymp_X)$ where $|X|$ is an at most countable set (the *web* of X), and $\overset{\frown}{\smile}_X, \asymp_X \subseteq P(|X|)$ with $\overset{\frown}{\smile}_X \cup \asymp_X = P(|X|)$. The set $\overset{\frown}{\smile}_X$ is called *coherence* and the set $\asymp_X$ is called *incoherence*. Their intersection is called *neutrality* (notation: N_X).

The strict coherence $\frown_X$ of X is the complementary set of $\asymp_X$ with respect to $P(|X|)$ and the strict incoherence $\smile_X$ is the complementary of $\overset{\frown}{\smile}_X$. Clearly, one can define a P-coherence space X by specifying two sets among $\overset{\frown}{\smile}_X$, $\asymp_X$, N_X, $\frown_X$ and $\smile_X$ subject to obvious constraints (for instance, one must have $\mathsf{N}_X \subseteq \overset{\frown}{\smile}_X$, $\smile_X \cap \frown_X = \emptyset \ldots$).

Definition 2.2 Let X be a P-coherence space. A *clique* of X is a subset x of $|X|$ such that $P(x) \subseteq \overset{\frown}{\smile}_X$. We denote by $\mathrm{Cl}(X)$ the set of all cliques of X.

We shall say that a P-coherence space X is reflexive if neutrality corresponds to equality in the following sense: for $x \subseteq |X|$, one has $P(x) \subseteq \mathsf{N}_X$ iff x is a singleton. This property is satisfied in the usual coherence and hypercoherence semantics.

From now on, we shall assume that a subset K of $\mathbb{N} \setminus \{0,1\}$ is given, and we call the corresponding $\mathcal{M}_K$-coherence space a K-coherence space.

The notation $|X|$ for the web of X might confuse the reader since we already used this notation for the relational interpretation of a formula. The confusion is on purpose since the web of the interpretation of a formula in the K-coherence model will be its relational interpretation.

2.3 Interpreting MALL... nothing new

The interpretation of the MALL fragment follows a standard pattern. We define directly the connectives of linear logic on K-coherence spaces (rather than defining by induction the interpretation of formulae).

Linear negation is the exchange of coherence and incoherence, that is $X^\perp = (|X|, \asymp_X, \overset{\frown}{\smile}_X)$.

Both additive constants are the empty K-coherence space ($0 = \top = (\emptyset, \emptyset, \emptyset)$). Both multiplicative constants are the reflexive one point K-coherence space $1 = \perp = (\{*\}, \mathcal{M}_K(\{*\}), \mathcal{M}_K(\{*\}))$.

Let X_1 and X_2 be two K-coherence spaces.

- Provided $|X_1|$ and $|X_2|$ are disjoint, $X_1 \oplus X_2$ is defined by setting $|X_1 \oplus X_2| = |X_1| \cup |X_2|$, $\mathsf{N}_{X_1 \oplus X_2} = \mathsf{N}_{X_1} \cup \mathsf{N}_{X_2}$ and $\frown_{X_1 \oplus X_2} = \frown_{X_1} \cup \frown_{X_2}$. And of course $X_1 \& X_2 = ({X_1}^{\perp} \oplus {X_2}^{\perp})^{\perp}$. If the webs of the two spaces involved are not disjoint, one uses the disjoint sum operation instead of the union.
- The space $X_1 \otimes X_2$ is defined as follows. We set $|X_1 \otimes X_2| = |X_1| \times |X_2|$. For $i = 1, 2$, let $\pi_i : \mathcal{M}_K(|X_1 \otimes X_2|) \to \mathcal{M}_K(|X_i|)$ be the canonical projections defined by $\pi_1([(a_i, b_i) \mid i \in I]) = [a_i \mid i \in I]$ and similarly for π_2. Then for each $s \in \mathcal{M}_K(|X_1 \otimes X_2|)$ we set

  $$\begin{aligned} s \in \mathsf{N}_{X_1 \otimes X_2} &\text{ iff } \pi_1(s) \in \mathsf{N}_{X_1} \text{ and } \pi_2(s) \in \mathsf{N}_{X_2} \\ s \in \smile_{X_1 \otimes X_2} &\text{ iff } \pi_1(s) \in \smile_{X_1} n \text{ or } \pi_2(s) \in \smile_{X_2} \end{aligned}$$

  which suffices to determine $\mathsf{N}_{X_1 \otimes X_2}, \smile_{X_1 \otimes X_2}$ and $\frown_{X_1 \otimes X_2}$. We also set $X_1 \parr X_2 = ({X_1}^{\perp} \otimes {X_2}^{\perp})^{\perp}$.

The *linear map* construction $\multimap$ between K-coherence spaces is defined by setting $X \multimap Y = X^{\perp} \parr Y$. A *linear morphism* from X to Y, two K-coherence spaces, is a K-clique of $X \multimap Y$. Remark that

$$s \in \mathrel{\overset{\frown}{\smile}}_{X \multimap Y} \text{ iff } \begin{cases} \pi_1(s) \in \mathrel{\overset{\frown}{\smile}}_X \implies \pi_2(s) \in \mathrel{\overset{\frown}{\smile}}_Y \\ \pi_1(s) \in \frown_X \implies \pi_2(s) \in \frown_Y \end{cases} \tag{1}$$

or equivalently,

$$s \in \mathrel{\overset{\frown}{\smile}}_{X \multimap Y} \text{ iff } \begin{cases} \pi_2(s) \in \smile_Y \implies \pi_1(s) \in \smile_X \\ (\pi_1(s) \in \mathrel{\overset{\frown}{\smile}}_X \text{ and } \pi_2(s) \in \mathsf{N}_Y) \implies \pi_1(s) \in \mathsf{N}_X. \end{cases} \tag{2}$$

We denote by $\mathbf{NCOH}_K$ the category whose objects are the K-coherence spaces, whose morphisms are the linear morphisms and where compositions and identities are defined as in **Rel**, that is

$$f \fatsemi g = \{(a, c) \mid \exists b, (a, b) \in f \text{ and } (b, c) \in g\}$$

and $\mathrm{id}_X = \{(a, a) \mid a \in |X|\}$. For every $K' \subseteq K$, the corresponding categories come naturally with forgetful functors $U_{K,K'} : \mathbf{NCOH}_K \to \mathbf{NCOH}_{K'}$ which act as the identity on morphisms.

The *boolean type*, denoted by **bool** and represented by the formula $1 \oplus 1$ will be interpreted, in $\mathbf{NCOH}_{\mathbb{N} \setminus \{0,1\}}$, by the reflexive $\mathbb{N} \setminus \{0, 1\}$-coherence space whose web is $\{\mathtt{t}, \mathtt{f}\}$ and whose coherence is $\mathcal{M}_{\mathbb{N} \setminus \{0,1\}}(\{\mathtt{t}\}) \cup \mathcal{M}_{\mathbb{N} \setminus \{0,1\}}(\{\mathtt{f}\})$.

Proposition 2.3 (model of MALL) *For each $K \subseteq \mathbb{N} \setminus \{0, 1\}$, the category $\mathbf{NCOH}_K$ is a model of* MALL *and for each $K' \subseteq K$ (in particular for $K' = \emptyset$) the functor $U_{K,K'}$ is logical w.r.t. the $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$* MALL *models (that is, commutes to the interpretations of sequents and proofs).*

This means that $\mathbf{NCOH}_K$ is a symmetric monoidal closed category (with $\otimes$ as tensor product and $\multimap$ as function space constructor) which is $*$-autonomous ($\perp$ being the dualizing object), and furthermore, has all finite products and coproducts (see [AC98] for precise definitions). The proof is a straightforward verification.

Remark 2.4 (foliation) *The coherence relation is* foliated *with respects to the interpretation of* MALL i.e. *for each formula A of* MALL *the coherence relations on multisets of cardinality n in the interpretation of A is totally determined by the coherence relations on multisets of cardinality n in the interpretation of the sub-formulae of A. In fact, this is exactly by constructing independently each coherence relation of level k for $k \in K$ in the Bucciarelli-Ehrhard machinery that the K-coherence spaces model has been obtained, so this remark also holds for the* LL *models with the exponential provided by this machinery. Anticipating a bit, it will also hold for the new exponential construction we present (and the forgetful functors $U_{K,K'}$ will still be logical in* LL*).*

2.4 Exponentials

Using the constructions presented by Bucciarelli and Ehrhard in [BE01], one can define exponentials for K-coherence spaces.

This gives a model which accepts the variant

$$\begin{aligned} f = \{ \ & (([], \quad [\mathtt{t},\mathtt{t}],\ [\mathtt{f},\mathtt{f}]),\ \mathtt{t}),\\ & (([\mathtt{f},\mathtt{f}],\ [], \quad [\mathtt{t},\mathtt{t}]),\ \mathtt{t}),\\ & (([\mathtt{t},\mathtt{t}],\ [\mathtt{f},\mathtt{f}],\ []), \quad \mathtt{t})\} \end{aligned}$$

of the well known Berry example of a stable and non sequential morphism from **bool** $\times$ **bool** $\times$ **bool** to **bool**. Of course the multiset based hypercoherence model[3] rejects such first order non-sequential functions.

The really surprising fact is that one can easily correct this by choosing another definition for the coherence relations of the exponential construction and obtain in that way a new model of linear logic. Among these variants for the exponentials there is a *most general one* in a sense which will be made precise in theorem 1 and corollary 1.

Definition 2.5 For each K-coherence space X we define the K-coherence space $!X$ as follow. Its web is $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and for each element $[x_i \mid i \in I]$ of $\mathcal{M}_K(|!X|)$ we set:

$$[x_i \mid i \in I] \in \smile_{!X} \text{ iff } \exists (a_i)_{i\in I}, [a_i \mid i \in I] \in \smile_X \text{ and } \forall i \in I, a_i \in x_i \quad (3)$$

and $[x_i \mid i \in I] \in \mathsf{N}_{!X}$ iff

$$[x_i \mid i \in I] \notin \smile_{!X} \text{ and } \exists (a_i^j)_{i\in I}^{j\in J}, \begin{cases} \forall i \in I, [a_i^j \mid j \in J] = x_i \\ \forall j \in J, [a_i^j \mid i \in I] \in \mathsf{N}_X \end{cases} \quad (4)$$

[3] The original exponentials of hypercoherences are set-based (their webs are sets of cliques), but can easily be adapted to a multiset-based setting.

We also define $?X$ by setting $?X = (!X^{\perp})^{\perp}$.

When $\forall i \in I, a_i \in x_i$ we say that $[a_i \mid i \in I]$ is a *section* of $[x_i \mid i \in I]$ and we write $[a_i \mid i \in I] \lhd [x_i \mid i \in I]$.

When $K = \emptyset$, the "of course" construction on objects is the standard exponential of **Rel**.

Example 1 *Consider the K-coherence space G with web $|G| = \{a, b, c\}$ and such that if $u \in \mathcal{M}_{fin}(|G|)$ then: $u \in \mathsf{N}_G$ iff $\mathrm{supp}(u)$ is a singleton, $u \in \frown_G$ iff $\sharp\,\mathrm{supp}(u) = 2$ and $u \in \smile_G$ iff $\mathrm{supp}(u) = \{a, b, c\}$. (The space G is in fact the sub-space of $\boldsymbol{bool}^3 \to \boldsymbol{bool}$ of web the variant of the Berry's example f above).*

Suppose $2 \in K$. All the sections of $[[a], [b, c]]$ are coherent in G moreover $[a]$ and $[b, c]$ have not the same cardinality. So $[[a], [b, c]] \in \frown_{!G}$. Now suppose $3 \in K$. Then $[[a], [b, c], [b, c]]$ admits the strictly incoherent section $[a, b, c]$ but $[[a], [a], [b, c]]$ not and so $[[a], [b, c], [b, c]] \in \smile_{!G}$ but $[[a], [a], [b, c]] \in \frown_{!G}$. So the coherence relations of $!G$ depends on multiplicities.

For each $k \in K$ such that $k \geq 3$, each $m \in \mathcal{M}_{\{k\}}(|G|)$ such that $\mathrm{supp}(m) = \{[a, b], [a, c]\}$ is strictly incoherent in $!G$ but if $2 \in K$, $[[a, b], [a, c]] \in \frown_{!G}$ (all the sections of $[[a, b], [a, c]]$ are coherent in G and b is not neutral with any element of $[a, c]$).

Finally $[[a, b, c], [a, b, c], [a, b, c]]$ is an example of a non neutral (strictly incoherent, here) multiset in $!G$ of support a singleton.

Proposition 2.6 (model of LL) *Any category $\mathbf{NCOH}_K$ with the exponentials of definition 2.5 is a model of* LL *(see [AC98] and [Bie95]) and for each $K' \subseteq K$ (in particular for $K' = \emptyset$) the functor $U_{K,K'}$ is logical w.r.t. the $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$* LL *models.*

Proof For the functoriality of $!$ as for its comonad structure we just follow the standard **Rel** construction. That is, for each morphism $f \in \mathbf{NCOH}_K(X, Y)$ we set:

$$!f = \{([a_1, \ldots, a_n], [b_1, \ldots, b_n]) \mid n \in \mathbb{N}, \forall i \in \{1, \ldots, n\}, (a_i, b_i) \in f\}$$

and for each K-coherence space X we set:

$$\begin{aligned} \mathrm{der}_X &= \{([a], a) \mid a \in |X|\} \quad \text{and} \\ \mathrm{dig}_X &= \{(\sum_{1 \leq i \leq n} x_i, [x_1, \ldots, x_n]) \mid n \in \mathbb{N}, [x_1, \ldots, x_n] \in |!!X|\}. \end{aligned}$$

To show that $(!, \mathrm{der}, \mathrm{dig})$ is a comonad of $\mathbf{NCOH}_K$, exploiting that all the required equalities (commutative diagrams) already hold in $\mathbf{NCOH}_\emptyset$, and therefore also in $\mathbf{NCOH}_K$, we only need to prove that $!f$ is a clique of $!X \multimap !Y$, that der_X is a clique of $!X \multimap X$ and that dig_X is a clique of $!X \multimap !!X$.

Let $[(x_j, y_j) \mid j \in J] \in \mathcal{M}_K(!f)$. If $[b_j \mid j \in J] \lhd [y_j \mid j \in J]$ then by construction of $!f$ there exists $[a_j \mid j \in J]$ such that $[(a_j, b_j) \mid j \in J] \in \mathcal{M}_K(f)$ and $[a_j \mid j \in J] \lhd [x_j \mid j \in J]$. Remark that since f is a clique, we have $[(a_j, b_j) \mid j \in J] \in \frown_{X \multimap Y}$. In particular, if $[b_j \mid j \in J] \in \smile_Y$ then $[a_j \mid j \in J] \in \smile_X$. Hence if $[y_j \mid j \in J]$ admits a strict incoherent section then $(x_j)_{j \in J}$ admits one too. So $[y_j \mid j \in J] \in \smile_{!Y} \implies [x_j \mid j \in J] \in \smile_{!X}$. Now suppose $[x_j \mid j \in J] \in \frown_{!X}$ and $[y_j \mid j \in J] \in \mathsf{N}_{!Y}$. We must prove that $[x_j \mid j \in J] \in \mathsf{N}_{!X}$. There exists $(b^i_j)_{(i,j) \in I \times J}$ such that $\forall j \in J, y_j = [b^i_j \mid i \in I]$ and $\forall i \in I, [b^i_j \mid j \in J] \in \mathsf{N}_Y$. By construction of $!f$ there exists $(a^i_j)_{(i,j) \in I \times J}$ such that $\forall (i,j) \in I \times J, (a^i_j, b^i_j) \in f$ and $\forall j \in J, x_j = [a^i_j \mid i \in I]$. Since $[x_j \mid j \in J] \in \frown_{!X}$ for each $i \in I$, $[a^i_j \mid j \in J] \in \frown_X$. But, for each $i \in I$, $[(a^i_j, b^i_j) \mid j \in J] \in \mathcal{M}_K(f) \subseteq \frown_{X \multimap Y}$ and $[b^i_j \mid j \in J] \in \mathsf{N}_Y$ so $[a^i_j \mid j \in J] \in \mathsf{N}_X$, for each $i \in I$. Finally $[x_j \mid j \in J] \in \mathsf{N}_{!X}$ which concludes the proof that $!f$ is a clique.

The fact that der_X is a clique is straightforward. We now prove that dig_X is a clique of $!X \multimap !!X$. Let $[(\sum_{i \in I_j} x^j_i, [x^j_i \mid i \in I_j]) \mid j \in J] \in \mathcal{M}_K(\mathrm{dig}_X)$.

Suppose $[[x^j_i \mid i \in I_j] \mid j \in J] \in \smile_{!!X}$. Then this multiset admits a section $[y_j \mid j \in J]$ strictly incoherent in $!X$. Hence this section $[y_j \mid j \in J]$ admits a section $[a_j \mid j \in J]$ strictly incoherent in X. Clearly this last section is also a section of $[\sum_{i \in I} x^j_i \mid j \in J]$ so this multiset is strictly incoherent in $!X$.

Now suppose $[\sum_{i \in I_j} x^j_i \mid j \in J] \in \frown_{!X}$ and $[[x^j_i \mid i \in I_j] \mid j \in J] \in \mathsf{N}_{!!X}$. Then there exists a family $(y^j_i)^{j \in J}_{i \in I}$ such that: for all $j \in J$, $[y^j_i \mid i \in I]$ equals $[x^j_i \mid i \in I_j]$ (so $I = I_j$ and $\sum_{i \in I} y^j_i = \sum_{i \in I_j} x^j_i$); and for all $i \in I$, $[y^j_i \mid j \in J] \in \mathsf{N}_{!X}$. Hence for each $i \in I$, there exists a family $(a^j_{i,l})^{j \in J}_{l \in L_i}$ such that for all $j \in J$, $y^j_i = [a^j_{i,l} \mid l \in L_i]$ and such that for all $l \in L_i$, $[a^j_{i,l} \mid j \in J] \in \mathsf{N}_X$. Without any lost of generalities the L_i can be chosen pairwise disjoint. Setting $L = \cup_{i \in I} L_i$, we have then $\sum_{l \in L} a^j_l = \sum_{i \in I_j} x^j_i$ and for all $l \in L$, $[a^j_l \mid j \in J] \in \mathsf{N}_X$. Hence $[\sum_{i \in I_j} x^j_i \mid j \in J] \in \mathsf{N}_{!X}$.

The set $\{([], *)\}$ is a clique of $!\top \multimap 1$ and the set $\{(*, [])\}$ is a clique of $1 \multimap !\top$ so $!\top \cong 1$. We now prove that $!(X \& Y) \cong !X \otimes !Y$, for each X and Y. The graph f of the bijection map

$$\begin{cases} \mathcal{M}_{\mathrm{fin}}(|X|) \times \mathcal{M}_{\mathrm{fin}}(|Y|) \to \mathcal{M}_{\mathrm{fin}}(|X \& Y|) \\ \qquad\qquad (x, y) \mapsto x + y \end{cases}$$

is a relational isomorphism. It remains to prove that f is a clique of the space $!(X \& Y) \multimap (!X \otimes !Y)$ and that its transpose is a clique of the space $(!X \otimes !Y) \multimap !(X \& Y)$. Consider a multiset $[((x_i, y_i), x_i + y_i) \mid i \in I] \in \mathcal{M}_K(f)$. Since an element of $\smile_{X \& Y}$ is either an element of $\smile_X$ or an element of $\smile_Y$, a section s of $[x_i + y_i \mid i \in I]$ is strictly incoherent in $X \& Y$ iff s is a strictly incoherent section of $[x_i \mid i \in I]$ or of $[y_i \mid i \in I]$. It follows that

$$[x_i + y_i \mid i \in I] \in \smile_{!(X \& Y)} \iff [(x_i, y_i) \mid i \in I] \in \smile_{!X \otimes !Y}.$$

An element of $\mathsf{N}_{X \& Y}$ is either an element of N_X or an element N_Y. Hence, if $[x_i + y_i \mid i \in I]$ is neutral in $!(X \& Y)$, there exists a family $(c^j_i)^{j \in J}_{i \in I}$ such that for each $j \in J$, $[c_i^j \mid i \in I] \in \mathsf{N}_{X\&Y}$ and such that $J = J_X + J_Y$ with, for each $i \in I$, $[c_i^j \mid j \in J_X] = x_i$ and $[c_i^j \mid j \in J_Y] = y_i$ and this family splits into two families, the first one corresponding to the neutrality of $[x_i \mid i \in I]$ in $!X$ and the other one to the neutrality of $[y_i \mid i \in I]$ in $!Y$. Consequently the neutrality of $[x_i + y_i \mid i \in I]$ in $!(X\&Y)$ implies the neutrality of $[(x_i, y_i) \mid i \in I]$ in $!X \otimes !Y$. The converse is straightforward. So the required isomorphisms $!\top \cong 1$ and $!(X \,\&\, Y) \cong !X \otimes !Y$ holds. At last we directly obtain that this two isomorphisms are naturals and that the adjonction involved by the comonad is monoidal (see [Bie95]) just by using the fact that this is already the case in **Rel**. □

3 The *of course* is the co-free commutative $\otimes$-comonoid

A commutative comonoid on a symmetric monoidal category $\mathcal{C}$, with respect to a monoidal structure $(\otimes, \mathrm{sym}, \mathrm{ass}, \mathrm{unit})$, is a 3-tuple $M = (\underline{M}, u_M, \mu_M)$ where $\underline{M} \in \mathcal{C}$, $u_M \in \mathcal{C}(\underline{M}, 1)$ and $\mu_M \in \mathcal{C}(\underline{M}, \underline{M} \otimes \underline{M})$ such that (associativity) $\mathrm{id}_{\underline{M}} \otimes \mu_M = (\mu_M \otimes \mathrm{id}_{\underline{M}}) \mathbin{;} \mathrm{ass}_{\underline{M},\underline{M},\underline{M}}$; (neutrality) $\mu_M \mathbin{;} (\mathrm{id}_{\underline{M}} \otimes u_M) = \mathrm{unit}_{\underline{M}}$ and (commutativity) $\mu_M \mathbin{;} \mathrm{sym}_{\underline{M}} = \mu_M$. A comonoid morphism f from (A, u_A, μ_A) to (B, u_B, μ_B) is a morphism $f \in \mathcal{C}(A, B)$ such that $f \mathbin{;} u_B = u_A$ and $f \mathbin{;} \mu_B = \mu_A \mathbin{;} (f \otimes f)$.

In each categorical model $\mathcal{C}$ of linear logic the "of course" naturally provides a commutative comonoid $(!X, \mathrm{weak}, \mathrm{cont})$ for each object X: weak is $!\top_X$ where $\top_X$ is the unique morphism of $\mathcal{C}(X, \top)$ and cont is $(!\langle \mathrm{id}_X, \mathrm{id}_X \rangle) \mathbin{;} e_X$ where $\langle \mathrm{id}_X, \mathrm{id}_X \rangle$ denotes the pairing of the identity with itself and where e_X is the isomorphism $!(X \,\&\, X) \cong !X \otimes !X$. For instance in $\mathbf{NCOH}_K$, weak $= \{([], *)\}$ and cont $= \{(x_1 + x_2, (x_1, x_2)) \mid x_1, x_2 \in |!X|\}$. Moreover for each $f \in \mathcal{C}(X, Y)$, $!f$ is a $\otimes$-comonoid morphism between $(!X, \mathrm{weak}, \mathrm{cont})$ and $(!Y, \mathrm{weak}_Y, \mathrm{cont}_Y)$.

A commutative comonoid (F, u_F, μ_F) is said to be co-free over an object X of $\mathcal{C}$ when there exists a morphism $d \in \mathcal{C}(F, X)$ such that for each (A, u_A, μ_A), and for each $f \in \mathcal{C}(A, X)$ there exists a unique comonoid morphism f_* from (A, u_A, μ_A) to (F, u_F, μ_F) such that $f_* \mathbin{;} d = f$.

By extension the "of course" ! is said to be the co-free commutative $\otimes$-comonoid or, for short, to be co-free, when for each commutative comonoid (A, u_A, μ_A), for each $X \in \mathcal{C}$ and for each $f \in \mathcal{C}(A, X)$ there exists a unique comonoid morphism $f_* : (A, u_A, \mu_A) \to (!X, \mathrm{weak}, \mathrm{cont})$ such that

$$f_* \mathbin{;} \mathrm{der}_X = f.$$

Remark that if ! is co-free then $f_* = \mathrm{id}_* \mathbin{;} !f$ where id is the identity morphism in $\mathcal{C}(A, A)$.

Lemma 1 *In* **Rel** *the exponential is co-free. Moreover if (A, u_A, μ_A) is a commutative $\otimes$-comonoid in* **Rel** *then $(a, x) \in (\mathrm{id}_A)_*$ iff if $(a_i)_{1 \le i \le n}$ is such that $[a_1, \ldots, a_n] = x$ then $\exists (b_i)_{0 \le i \le n}$ such that $b_0 = a$, $(b_i, (a_{i+1}, b_{i+1})) \in \mu_A$*

for each $i < n$, and $(b_n, *) \in u_A$.

Theorem 1 (co-free) *The "of course" $!$ is the co-free commutative $\otimes$-comonoid of $\mathbf{NCOH}_K$ and the forgetful functor $U_{K,\emptyset} : \mathbf{NCOH}_K \to \mathbf{Rel}$ maps this structure to the standard one.*

Proof We prove that for each commutative comonoid (A, u_A, μ_A) of $\mathbf{NCOH}_K$ for each $X \in \mathbf{NCOH}_K$ and for each $f \in \mathbf{NCOH}_K(A, X)$, there exists a unique comonoid morphism $f_* : (A, u_A, \mu_A) \to (!X, \mathrm{der}, \mathrm{cont})$ such that $f_* \mathbin{;} \mathrm{der} = f$.

But if there is such an f_* in $\mathbf{NCOH}_K$ then $U_{K,\emptyset}(f_*)$ is a comonoid morphism $(U_{K,\emptyset}(A), U_{K,\emptyset}(u_A), U_{K,\emptyset}(\mu_A)) \to (U_{K,\emptyset}(!X), U_{K,\emptyset}(\mathrm{der}_X), U_{K,\emptyset}(\mathrm{cont}))$ and $U_{K,\emptyset}(f_*) \mathbin{;} U_{K,\emptyset}(\mathrm{der}_X) = U_{K,\emptyset}(f)$. As $(U_{K,\emptyset}(!X), U_{K,\emptyset}(\mathrm{der}_X), U_{K,\emptyset}(\mathrm{cont}))$ is the co-free $\otimes$-comonoid in $\mathbf{Rel}$ this means that $U_{K,\emptyset}(f_*)$ is equal to $U_{K,\emptyset}(f)_*$ in $\mathbf{Rel}$. Moreover $U_{K,\emptyset}(!f) = !U_{K,\emptyset}(f)$, and $U_{K,\emptyset}(f)_* = \mathrm{id}_* \mathbin{;} !U_{K,\emptyset}(f)$. So the only thing to prove is $\mathrm{id}_* \in \mathrm{Cl}(A \multimap !A)$. Let $[(a^i, [a^i_1, \dots, a^i_{n_i}]) \mid i \in I]$ be an element of $\mathcal{M}_K(\mathrm{id}_*)$. Then, using lemma 1, for each $i \in I$, let $(b^i_j)_{0 \le j \le n_i}$ be a family such that $b^i_0 = a^i$, $(b^i_j, (a^i_{j+1}, b^i_{j+1})) \in \mu_A$ for each $j < n_i$, and $(b^i_{n_i}, *) \in u_A$.

Suppose $[[a^i_1, \dots, a^i_{n_i}] \mid i \in I] \in \smile_{!A}$ then this multiset admits a strict incoherent section. Up to a choice of an adequate indexation of the multiset $[a^i_1, \dots, a^i_{n_i}]$, we can suppose without any loss of generality that this section is $[a^i_1 \mid i \in I]$. Remark that due to the existence of a section, none of the n_i is zero. We then have $[(a^i, (a^i_1, b^i_1)) \mid i \in I] \in \mathcal{M}_K(\mu_A)$ with $[a^i_1 \mid i \in I] \in \smile_A$. Hence $[(a^i_1, b^i_1) \mid i \in I] \in \smile_{A \otimes A}$. And, since $[(a^i, (a^i_1, b^i_1)) \mid i \in I]$ must be coherent for μ_A to be a clique of $A \multimap (A \otimes A)$, we then have $[a^i \mid i \in I] \in \smile_A$.

Now suppose $[a^i \mid i \in I] \in \frown_A$ and $[[a^i_1, \dots, a^i_{n_i}] \mid i \in I] \in \mathsf{N}_A$. According to the definition of neutrality in the "of course", all the n_i are equal, say $n_i = n (\forall i \in I)$, and, up to an appropriate re-indexing, $[a^i_j \mid i \in I] \in \mathsf{N}_A$, for each $1 \le j \le n$. Since $[(b^i_n, *) \mid i \in I] \in \mathcal{M}_K(u_A) \subseteq \frown_{A \multimap 1}$ and $[* \mid i \in I] \in \mathsf{N}_1$, this means that $[b^i_n \mid i \in I] \in \asymp_A$. Now suppose $[b^i_{k+1} \mid i \in I] \in \asymp_A$ for a certain $k < n$, then using $[a^i_{k+1} \mid i \in I] \in \mathsf{N}_A$ and $[(b^i_k, (a^i_{k+1}, b^i_{k+1})) \mid i \in I] \in \mathcal{M}_K(\mu_A)$ it follows that $[b^i_k \mid i \in I] \in \asymp_A$ thus for all $j \le n$, $[b^i_j \mid i \in I] \in \asymp_A$ and in particular $[b^i_0 \mid i \in I] = [a^i \mid i \in I]$ then proved to be both coherent and incoherent, that is to be neutral. So id_* is a clique. $\square$

Consider a sub-category $\mathcal{C}$ of $\mathbf{NCOH}_K$ which is a categorical model of intuitionistic multiplicative exponential linear logic [4]. Let E be the operation modeling the "of course" on objects in $\mathcal{C}$. We shall say that this model is *multiset based* if for each $X \in \mathcal{C}$:

- the web of $E(X)$ is made of multisets of points of the web of X ;
- the commutative comonoid structure provided with $E(X)$ by the model is

[4] We do not require $\mathcal{C}$ satisfies more, but a typical $\mathcal{C}$ for our purpose will be a new Seely category where the multiplicative additive and orthogonal constructions are the ones of $\mathbf{NCOH}_K$ and so one should have a model for the full linear logic fragment, where the exponentials are given by a comonad.

defined by $\text{weak}'_X = \{([], *)\}$ (of type $E(X) \to 1$) and

$$\text{cont}'_X = \{(x_1 + x_2, (x_1, x_2)) \mid x_1 + x_2 \in |E(X)| \text{ and } x_1, x_2 \in \mathcal{M}_{\text{fin}}(|X|)\}$$

(of type $E(X) \to E(X) \otimes E(X)$);

- the associated dereliction morphism is $\text{der}'_X = \{([a], a) \mid a \in |X|\}$ (of type $E(X) \to X$).

Corollary 1 (maximality of the co-free "of course") *If a sub-monoidal category $\mathcal{C}$ of $\mathbf{NCOH}_K$ is a multiset based* LL *model, of "of course" E then, for each object $X \in \mathcal{C}$,*

$$\smile_{E(X)} \subseteq \smile_{!X} \tag{5}$$
and
$$\frown_{E(X)} \subseteq \frown_{!X}. \tag{6}$$

"Sub-monoidal category" means that $\mathcal{C}$ is a sub-category of $\mathbf{NCOH}_K$ equipped with the same symmetric monoidal structure as $\mathbf{NCOH}_K$.

Proof Since $\mathcal{C}$ is a model of LL, $E(X)$ comes with a $\otimes$-comonoid structure $(E(X), \text{weak}'_X, \text{cont}'_X)$ where weak'_X is the weakening morphism and cont'_X is the contraction morphism. Let der'_X be the dereliction morphism for X of $\mathcal{C}$. Using theorem 1, there exists a morphism $\text{der}'_{X,*}$ of $E(X) \multimap !X$. Using lemma 1 and due to the fact that $\mathcal{C}$ is multiset based we obtain that $\text{der}'_{X,*}$ is equal to $\{(x, x) \mid x \in |E(X)|\}$ (the inclusion morphism of $E(X)$ in $!X$). Finally, using (1), it comes (5) and (6). □

We shall say that a multiset based model of LL in a sub-category $\mathcal{C}$ of $\mathbf{NCOH}_K$ is *non-uniform* when the web of the "of course" E is the whole set of finite multisets (*i.e.* $|E(X)| = \mathcal{M}_{\text{fin}}(|X|), \forall X \in \mathcal{C}$).

Corollary 2 (sequentiality failure)
Each non-uniform multiset based model of LL *in a sub-monoidal category $\mathcal{C}$ of $\mathbf{NCOH}_K$ fails to reject the morphism $\{([\mathit{t}], \mathit{t}), ([\mathit{f}], \mathit{t}), ([\mathit{t}, \mathit{f}], \mathit{t})\}$ of type* ***bool*** $\to$ ***bool****.*

This is a strong negative results since this set cannot be included in the interpretation of a term of PCF. Our sentiment is that it will be the same for any reasonably sequential calculus interpretable in our model of linear logic. Remark that the very similar morphism $\{([\mathtt{t}, \mathtt{t}], \mathtt{t}), ([\mathtt{f}, \mathtt{f}], \mathtt{t}), ([\mathtt{t}, \mathtt{f}], \mathtt{t})\}$ is the interpretation of $\lambda b.\, \mathtt{if}\ b\ \mathtt{then}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{t}\ \mathtt{else}\ \mathtt{t})\ \mathtt{else}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{t}\ \mathtt{else}\ \mathtt{t})$.

4 Relating uniform and non-uniform semantics

Definition 4.1 For each $K \subseteq \mathbb{N}\backslash\{0, 1\}$, let $\mathbf{NCoh}_K$ be the full sub-category of $\mathbf{NCOH}_K$ whose objects are the K-coherence spaces X such that

$$\mathsf{N}_X \subseteq \cup_{a \in |X|} \mathcal{M}_K\{a\} \tag{7}$$

Clearly $\mathbf{NCoh}_K$ is closed under the orthogonal, additive and multiplicative constructions. This is also the case for the exponential construction as easily verified. Indeed assume X has the property (7) and consider a neutral multiset $[x_i \mid i \in I]$ in $!X$. Then there exists a family $(a_i^j)_{i\in I}^{1\leq j\leq p}$ such that, for each $i \in I$, $x_i = [a_i^j \mid 1 \leq j \leq p]$ and, for each $1 \leq j \leq p$, $[a_i^j \mid i \in I] \in \mathsf{N}_X$. So using property (7) of X we obtain that there exists a family $(a^j)_{1\leq j\leq p}$ such that $a_i^j = a^j (\forall i, j)$ and consequently all the x_i are equal.

Hence this sub-category is a denotational model of propositional linear logic. Each forgetful functor $U_{K,K'}$ between $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$ (for $K' \subseteq K$) defines a forgetful functor between $\mathbf{NCoh}_K$ and $\mathbf{NCoh}_{K'}$ having similar properties and for which we use the same notation $U_{K,K'}$.

Proposition 4.2 (determinism) *If $X \in \mathbf{NCoh}_K$ and if x is a clique of X and y is an* anti-clique *of X (that is a clique of $X^\perp$) then $\sharp(x \cap y) \leq 1$.*

Proof Since $\mathcal{M}_K(x) \subseteq \frown_X$ and $\mathcal{M}(y) \subseteq \asymp_X$, $\mathcal{M}_K(x \cap y) \subseteq \mathsf{N}_X$ and we conclude using property (7). □

In fact this property can be made more precise since only certain points can be at the intersection of a clique and an anti-clique. These points constitute the *neutral web*.

Definition 4.3 Let $X \in \mathbf{NCoh}_K$. We call *neutral web* of X and we denote by $|X|_{\mathsf{N},K}$ (or simply by $|X|_{\mathsf{N}}$) the set $\{a \in |X| \mid \mathcal{M}_K(\{a\}) \subseteq \mathsf{N}_X\}$.

Example 2 *For the K-coherence space G of the example 1 page 10 we have: $[a, b] \in |!G|_{\mathsf{N},K}$, if $K \subseteq \{2\}$, $[a, b, c] \in |!G|_{\mathsf{N},K}$ and elsewhere $[a, b, c] \notin |!G|_{\mathsf{N},K}$.*

A key result about the neutral web is its behaviour when an "of course" construction is performed:

Lemma 2 (key lemma) *For $X \in \mathbf{NCoh}_K$ one has*

$$|!X|_{\mathsf{N},K} = \{x \in \mathcal{M}_{\mathit{fin}}(|X|_{\mathsf{N},K}) \mid \mathrm{supp}(x) \in \mathrm{Cl}(X)\}$$

Proof Let $x \in |!X|_{\mathsf{N},K}$. Then for all $k \in K$, there exists a family $(a_i^j)_{1\leq i\leq k}^{j\in J}$ such that $[a_i^j \mid j \in J] = x$ and $[a_i^j \mid 1 \leq i \leq k] \in \mathsf{N}_X$. Due to (7), for each $j \in J$, $a_1^j = \ldots = a_k^j$. Hence for all $k \in K$, for all $a \in x$, $k.[a] \in \mathsf{N}_X$. So $\mathrm{supp}(x) \subseteq |X|_{\mathsf{N},K}$. Each $y \in \mathcal{M}_K(\mathrm{supp}(x))$ is a section of the multiset $(\sharp y).[x] \in \mathsf{N}_{!X} \subseteq \frown_{!X}$, hence $\mathrm{supp}(x)$ is a clique. Thus the left to right inclusion is proved. Conversely, let $x \in \mathcal{M}_{\mathrm{fin}}(|X|_{\mathsf{N},K})$. If $\mathrm{supp}(x)$ is a clique then $k.[x] \in \frown_{!X}$ for any $k \in K$. Moreover each of the element a of x satisfies $k.[a] \in \mathsf{N}_X$ thus $k.[x] \in \frown_{!X}$ for any $k \in K$. And this proves the right to left inclusion. □

Example 3 *In $(!G)^\perp$, the set $x = \{[a, b], [a, c]\} \subseteq |(!G)^\perp|_{\mathsf{N},K}$ is not a clique if $2 \in K$ but is a clique if $2 \notin K$. Hence $[[a, b], [a, c]] \notin |!(!G)^\perp|_{\mathsf{N},\{2\}}$ and $[[a, b], [a, c]] \in |!(!G)^\perp|_{\mathsf{N},\{3\}}$.*

This lemma has many consequences.

Definition 4.4 If $X \in \mathbf{NCoh}_K$, the *neutral restriction* of X is the *sub-space* of X of web $|X|_{\mathsf{N}}$, that is $(|X|_{\mathsf{N}}, \mathsf{N}_X \cap M, \frown_X \cap M, \smile_X \cap M)$ where $M = \mathcal{M}_K(|X|_{\mathsf{N}})$, and the neutral restriction of a clique x of X is $x \cap |X|_{\mathsf{N}}$. The functor $N_K : \mathbf{NCoh}_K \to \mathbf{NCoh}_K$, sometimes simply denoted by N, associates to objects and morphisms their neutral restrictions.

One easily verifies that N_K is indeed a functor.

Proposition 4.5 *The functor N_K commutes with all the multiplicative additive constructions. Moreover $N_K! = N_K!N_K$.*

Proof The first statement is an obvious consequence of the corresponding definitions.

On objects, $N_K! = N_K!N_K$ is a consequence of lemma 2. Indeed, in the right part of the equality stated in this lemma, $\mathrm{Cl}(X)$ can be replaced with $\mathrm{Cl}(N_K X)$ since $\mathrm{supp}(x) \subseteq |X|_{\mathsf{N},K}$. This gives $|!X|_{\mathsf{N},K} = |!N_K X|_{\mathsf{N},K}$ which is what we wanted. The equality $N_K! = N_K!N_K$ on morphism is a straightforward consequence of the equality on objects. $\square$

Particular K-coherence spaces are the ones whose web coincide with their neutral web. They form a full sub-category of $\mathbf{NCoh}_K$.

Definition 4.6 Let $K \subseteq \mathbb{N}\backslash\{0,1\}$. A *uniform K-coherence space* is just a K-coherence space where neutrality coincides with equality: *i.e.* such that $\mathsf{N}_X = \cup_{a\in|X|}\mathcal{M}_K(\{a\})$. We denote by $\mathbf{Coh}_K$ the full sub-category of $\mathbf{NCoh}_K$ whose objects are the uniform K-coherence spaces.

Remark that uniform K-coherence spaces are particular reflexive P-coherence spaces. In a uniform K-coherence space X, any of the relations $\frown\!\!\!\!\smile_X$, $\frown_X$, $\asymp_X$, $\smile_X$ determines the whole structure of the space. A uniform $\{2\}$-coherence space is just an ordinary coherence space.

The functor N_K maps $\mathbf{NCoh}_K$ to $\mathbf{Coh}_K$ and on $\mathbf{Coh}_K$, N_K acts like the identity functor.

Additive and multiplicative constructions of $\mathbf{NCoh}_K$ preserve uniform K-coherence spaces. This is not the case for the "of course" functor. Fortunately, lemma 2 gives a clear hint on what should be the right exponentials for $\mathbf{Coh}_K$.

Definition 4.7 We define the functor $\underset{u}{!}$ interpreting the "of course" in $\mathbf{Coh}_K$ by setting $\underset{u}{!} = N_K!$. We denote by $\underset{u}{?}$ the corresponding "why not" functor.

The web of $\underset{u}{!}X$, called the *uniform web*, is then

$$|\underset{u}{!}X| = \{x \in \mathcal{M}_{\mathrm{fin}}(|X|) \mid \mathrm{supp}(X) \in \mathrm{Cl}(X)\}$$

and the coherence of $\underset{u}{!}X$ is then given by

$$M \in \frown\!\!\!\!\smile_{\underset{u}{!}X} \text{ iff } \{m \mid m \lhd M\} \subseteq \frown\!\!\!\!\smile_X.$$

This definition of the exponentials appears as a multiplicities aware version of the hypercoherences exponentials that have been introduced in [Ehr93].

As stated by the following theorem, these definitions give rise to a new class of uniform models together with a straightforward way to extract these interpretations from the non-uniform ones.

Theorem 2 *For each $K \subseteq \mathbb{N}\backslash\{0,1\}$, $\mathbf{Coh}_K$ equipped with the uniform exponentials and the standard multiplicative additive structures of $\mathbf{NCoh}_K$ is a categorical model of linear logic. Moreover:*

(i) *the functor $N_K : \mathbf{NCoh}_K \to \mathbf{Coh}_K$ is* logical *which means in particular that the* neutral restriction *of the K-coherence space $[A]_K$ is the uniform K-coherent interpretation $[A]^u_K$ of a formula A and that the neutral restriction $[\pi]_K \cap |\vdash \Gamma|_{\mathsf{N},K}$ of the K-coherence interpretation of a proof π of a sequent $\vdash \Gamma$ is the uniform K-coherence interpretation $[\pi]^u_K$ of π;*

(ii) *when $K = \{2\}$ this model is exactly the usual multiset based coherence model.*

Proof The multiplicative-additive part of the verification of the fact that $\mathbf{Coh}_K$ is a model of linear logic is easy and relies essentially on the fact that N commutes to all the additive and multiplicative constructions.

The exponential part is not very complicated either. By setting $\mathrm{der}_{u,X} = N(\mathrm{der}_X)$ and $\mathrm{dig}_{u,X} = N(\mathrm{dig}_X)$ for each $X \in \mathbf{Coh}_K$, we obtain two natural transformations $\mathrm{der}_u : N! \dot{\to} N\,\mathrm{id}$ and $\mathrm{dig}_u : N! \dot{\to} N!!$ in $\mathbf{Coh}_K$.

But N is the identity functor on $\mathbf{Coh}_K$, $N! = \underset{u}{!}$ and using proposition 4.5 ($N! = N!N$ in $\mathbf{NCoh}_K$) we obtain $N!! = \underset{u}{!}\,\underset{u}{!}$, and also $N!!! = \underset{u}{!}\,\underset{u}{!}\,\underset{u}{!}$. So der_u and dig_u are in fact natural transformations $\mathrm{der}_u : \underset{u}{!} \dot{\to} \mathrm{id}$ and $\mathrm{dig}_u : \underset{u}{!} \dot{\to} \underset{u}{!}\,\underset{u}{!}$.

These two natural transformations endow $\underset{u}{!}$ with a comonad structure. In fact we deduce the commutation of the required diagrams from the commutation of the corresponding diagrams already holding for the comonad $(!, \mathrm{der}, \mathrm{dig})$ by use of the functor N. The only non-obvious step is then to prove that for each $X \in \mathbf{Coh}_K$, $N\,\mathrm{dig}_{!X} = \mathrm{dig}_{u,\underset{u}{!}X}$ and $N\,\mathrm{der}_{!X} = \mathrm{der}_{u,\underset{u}{!}X}$. This can be done as follow. For all $f \in \mathbf{NCoh}_K(N!X, !X)$ one has $!f \mathbin{;} \mathrm{dig}_{!X} = \mathrm{dig}_{N!X} \mathbin{;} f$ hence $N(!f \mathbin{;} \mathrm{dig}_{!X}) = N(\mathrm{dig}_{!N!X} \mathbin{;} f)$ and so $N!f \mathbin{;} N\,\mathrm{dig}_{!X} = N(\mathrm{dig}_{N!X}) \mathbin{;} Nf$. The set $\mathrm{id}_{N!X}$ is clearly a clique of $N!X \multimap !X$ and so can be seen as a morphism i, the inclusion, from $N!X$ into $!X$. We then have the set equalities $N!i = \mathrm{id}_{\underset{u}{!}\,\underset{u}{!}X}$ and $Ni = \mathrm{id}_{\underset{u}{!}X}$ so finally by taking $f = i$ in the equation $N!f \mathbin{;} N\,\mathrm{dig}_{!X} = N(\mathrm{dig}_{N!X}) \mathbin{;} Nf$ we obtain the set equality $N\,\mathrm{dig}_{!X} = N\,\mathrm{dig}_{N!X}$ that is $N\,\mathrm{dig}_{!X} = \mathrm{dig}_{u,\underset{u}{!}X}$. Starting from the equation $N(\mathrm{der}_{!X} \mathbin{;} i) = N(!i \mathbin{;} \mathrm{der}_{!X})$ one proves $N\,\mathrm{der}_{!X} = \mathrm{der}_{u,\underset{u}{!}X}$ in the same way.

Using proposition 4.5 we obtain the isomorphisms $\underset{u}{!}A \otimes \underset{u}{!}B \cong \underset{u}{!}(A\&B)$ and $\underset{u}{!}\top \cong 1$.

$\mathbf{Coh}_K$ has been proved to be a categorical model of linear logic and there is nothing more to say for stating that N is logical.

The comonoid structure of the exponential $\underset{u}{!}$ is then the image of the comonoid structure of the exponential ! of $\mathbf{NCoh}_K$ through the functor N. The fact that $(\underset{u}{!}, \mathrm{der}_u)$ is co-free relies essentially on the set equality $\mathrm{der}_{u,X} = \mathrm{der}_X$ ($\forall X \in \mathbf{Coh}_K$) which is just a consequence of the fact that all singletons are K-cliques in $\mathbf{Coh}_K$. In fact, given a commutative comonoid (A, u_A, μ_A) of $\mathbf{Coh}_K$ and $f \in \mathbf{Coh}_K(A, X)$ one has $N(f_*) \,⨟\, \mathrm{der}_{u,X} = N(f_* \,⨟\, \mathrm{der}_X)$ for each $f \in \mathbf{Coh}_K(A, X)$ where f_* is the unique comonoid morphism $A \to !X$ such that $f_* \,⨟\, \mathrm{der}_X = f$. But $N(f_*) : A \to \underset{u}{!}\, X$ is also a comonoid morphism. Remark that the inclusion morphism $i : \underset{u}{!}\, X \to !X$ is a comonoid morphism hence $N(f_*) \,⨟\, i : A \to !X$ is a comonoid morphism. We also have the set equalities $N(f_*) \,⨟\, i = N(f_*)$ and, due to $\mathrm{der}_X = \mathrm{der}_{u,X}$, $N(f_*) \,⨟\, i \,⨟\, \mathrm{der}_X = N(f_*) \,⨟\, \mathrm{der}_{u,X} = f$. By uniqueness of f_*, $N(f_*) \,⨟\, i = f_*$, so we finally obtain the set equality $f_* = N(f_*)$, and the co-freeness of $\underset{u}{!}$ follows.

Finally, $[x, y] \in \bigcirc_{\underset{u}{!} X}$ iff $\forall a \in x, \forall b \in y, [a, b] \in \bigcirc_X$ that is, in $\mathbf{Coh}_{\{2\}}$, iff $\mathrm{supp}(x + y)$ is a clique. So in $\mathbf{Coh}_{\{2\}}$ which is the category of coherence spaces, $\underset{u}{!}$ is the well-known multiset based exponential of coherence spaces. □

We call the categorical model based on $\mathbf{Coh}_{\mathbb{N}\setminus\{0,1\}}$ the *multicoherence model*[5], we call *multicoherences* its objects, and we also call *non-uniform multicoherences* the objects of $\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$. The only difference between hypercoherences and multicoherences is that multicoherences take into account the multiplicity of points for the coherence relation.

Proposition 4.8 (sequentiality) *In the multicoherence model, every finite clique of function type* $!(\boldsymbol{bool} \,\&\, \ldots \,\&\, \boldsymbol{bool}) \multimap \boldsymbol{bool}$ *is sub-definable in* PCF.

Proof The proof follows the same scheme as for the usual hypercoherence model. □

Remark 4.9 *All cliques in the multicoherence model are cliques in the coherence model (this is a consequence of the foliation property).*

Hypercoherences are particular multicoherences: the multicoherences X such that

$$\forall u \in \bigcirc_X, \mathrm{supp}^{-1}(\mathrm{supp}(u)) \subseteq \bigcirc_X.$$

If X is a non-uniform multicoherence having this property for both the coherence relation and the incoherence relation we say that X is a *non-uniform hypercoherence* (So a non-uniform hypercoherence is indeed a $\mathcal{P}^*_{\mathrm{fin}}$-space).

If X is a non-uniform multicoherence, $S(X)$ is the non-uniform hyperco-

[5] General graph theory misses a term for such graphs and, contrarily to the *hyper-* situation where hypercoherences and hypergraphs are the same, *multigraphs* already exist but are not multicoherences.

herence defined by

$$\begin{aligned}\frown_{S(X)} &= \{u \in \mathcal{M}_{\mathbb{N}\setminus\{0,1\}}(|X|) \mid \mathrm{supp}^{-1}(\mathrm{supp}(u)) \subseteq \frown_X\}\\ \mathsf{N}_{S(X)} &= \{u \in \mathcal{M}_{\mathbb{N}\setminus\{0,1\}}(|X|) \mid \mathrm{supp}^{-1}(\mathrm{supp}(u)) \subseteq \mathsf{N}_X\}\end{aligned}$$

Remark that the operation $S\,\underset{u}{!}$ which maps X to $S(\underset{u}{!}\,X)$ is the hypercoherences multiset based "of course" construction on objects.

Theorem 3

(i) *The sub-category* **NHc** *of* $\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$ *of objects the non-uniform hypercoherences, equipped with the of course $S!$ on objects and acting like $!$ on morphisms is a model of linear logic.*

(ii) *The functor N from* **NHc** *to* **Hc**, *the category of hypercoherences, is logical (for the multiset based hypercoherences model).*

(iii) *The exponentials $S!$ and $S\,\underset{u}{!}$ are respectively co-free in* **NHc** *and* **Hc**.

Proof The proof of these statements follows from the proofs of proposition 2.6, theorem 1 and theorem 2. Just remark that some results can be re-used since for each non-uniform multicoherence X, one has $S(!X) = S(!S(X))$ and $N(S(X)) = S(N(X))$. □

Remark that corollary 1 applies to $S!$ and $S\,\underset{u}{!}$.

Example 4 *The K-coherence space G of the example 1 page 10 is uniform. Moreover when $K = \mathbb{N}\setminus\{0,1\}$, G is an hypercoherence. The multisets $[a,b]$ and $[a,c]$ are elements of $|!G|_{\mathsf{N}}$. The set $x = \{[a,b],[a,c]\} \subseteq |!G|_{\mathsf{N}}$ is a anti-clique of $S(\underset{u}{!}\,G)$. But this set is not an anti-clique (nor a clique) of $\underset{u}{!}\,G$. Hence each finite multiset of support x is an element of $|?S(\underset{u}{!}\,G)|_{\mathsf{N}}$ but not an element of $|?\,\underset{u}{!}\,G|_{\mathsf{N}} = |?!G|_{\mathsf{N}}$.*

Consider the situation where a same symmetric monoidal closed category has two different exponentials defining two different models of linear logic.

P.-A. Melliès has shown that if there is a *coercion* between the two "of course" which preserves some structure then the two models will have the same extensional collapse ([Mel01]).

This result easily applies to our situation. We then obtain that the multiset based coherence model and the non-uniform coherence space model have the same extensional collapse which is the set based coherence space model; the same for hypercoherences; and the same for multicoherences which we equip with the set based exponential $\underset{s}{!}$ defined by:

$$\begin{aligned}|\underset{s}{!}\,X| &= \{x \in \mathcal{P}_{\mathrm{fin}}(|X|) \mid x \in \mathrm{Cl}(X)\}\\ M \in \frown_{\underset{s}{!}X} &\text{ iff } \{m \mid m \lhd M\} \subseteq \frown_X.\end{aligned}$$

Of course this last exponential also provides a model of linear logic.

Finally using the fact that the set based hypercoherence and multicoherence models are both extensional we show that hypercoherences and multicoherences are extensionally different by exhibiting a relation at a functional type which is a clique in one of two models but not in the other:

Example 5 *For the hypercoherence G of our lasts examples above, one has that $\{a,b\}$ and $\{c\}$ are elements of $|\underset{s}{!}G| = |S(\underset{s}{!}G)|$. Moreover $(\{a,b\}, \boldsymbol{t})$ and $(\{c\}, \boldsymbol{f})$ are elements of $|\underset{s}{!}G \multimap \boldsymbol{bool}| = |S(\underset{s}{!}G) \multimap \boldsymbol{bool}|$. The relation $F = \{(\{a,b\}, \boldsymbol{t}), (\{c\}, \boldsymbol{f})\}$ is a clique of the hypercoherence $S(\underset{s}{!}G) \multimap \boldsymbol{bool}$. But F is not a clique of the multicoherence $\underset{s}{!}G \multimap \boldsymbol{bool}$, since $[\{a,b\},\{c\}]$ is coherent but $[\boldsymbol{t}, \boldsymbol{f}]$ is strictly incoherent.*

An exciting question is now to determine what sort of higher order sequentiality comes with multicoherences.

Another open question is: can the general construction Bucciarelli and Ehrhard introduced in [BE01] be modified so as to directly obtain the co-free exponentials in a general way?

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