Projecting games on hypercoherences

polarized bordered games

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Games and hypercoherences

Denotational semantics:

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(two-players plays)	(points/vertices)

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Comparison?

First answer: extensional collapse

Sequential algorithms **Hypercoherences**

strategies —— extensional collapse, Ehrhard 1999 ——→ cliques

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Quotient by partial equivalence relations (Kreisel 50′):

$$x \sim_\iota x \quad \text{and } f \sim_{\sigma\to\tau} g \text{ iff } x \sim_\sigma y \implies f(x) \sim_\tau g(y)$$

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- Limited to simple types ($\sigma := \iota \mid \sigma \to \sigma$).

Another bridge: direct projection

Games — time-forgetting projection → **Hypercoherences**

plays — projection → points

Another bridge: direct projection

Games strategies —— time-forgetting projection ——→ **Hypercoherences** sets of points

in general, the image of a strategy is not a clique

Another bridge: direct projection

Games strategies —— time-forgetting projection ——→ **Relational model** relations

relational model ≈ hypercoherences without coherence

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Games strategies —time-forgetting projection→ **Relational model** relations

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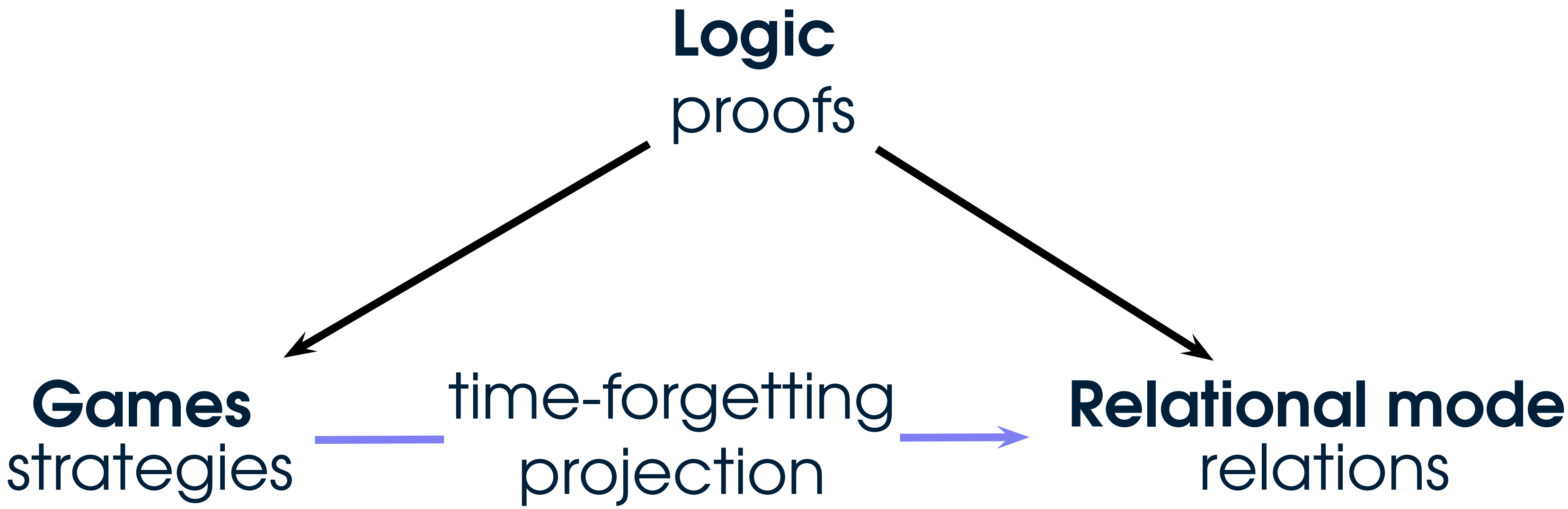

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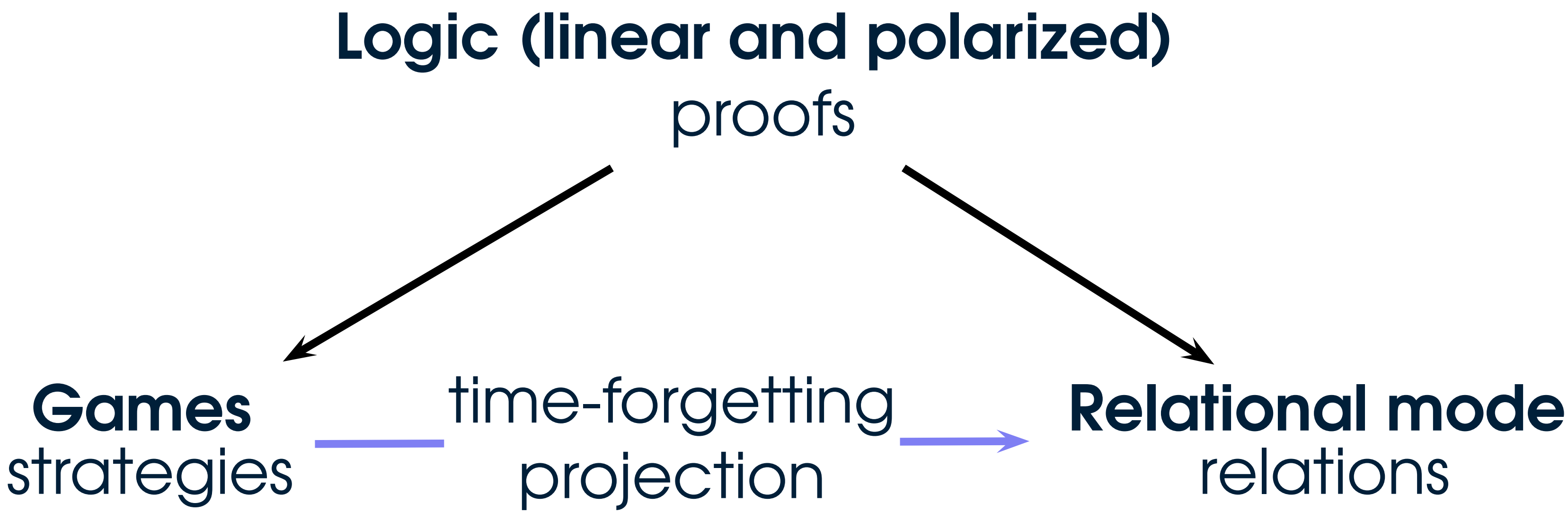

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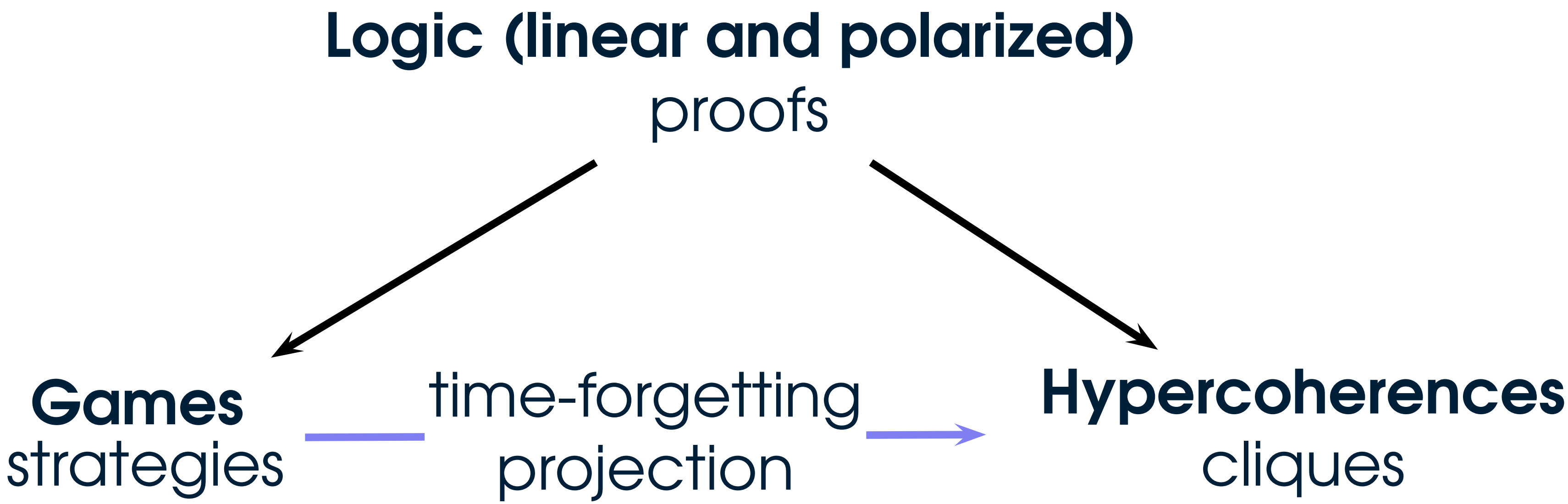

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Linear logic with polarities (Laurent)

positives $P := 0 \mid 1 \mid \alpha^\perp \mid P \oplus P \mid P \otimes P \mid !N$

negatives $N := \top \mid \bot \mid \alpha \mid N \,\&\, N \mid N \parr N \mid ?P$

$$\frac{}{\vdash \alpha, \alpha^\perp}\,(ax.) \qquad \frac{}{\vdash 1}\,(one) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\,(bot) \qquad \frac{}{\vdash \top, \Gamma}\,(top)$$

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Polarized bordered games

Games:

positive	$A = (+, A^o, A^p, S_A)$	$S_A \subseteq (A^p \cdot A^o)^*$
negative	$A = (-, A^o, A^p, S_A)$	$S_A \subseteq (A^o \cdot A^p)^*$

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Strategy $\sigma \subseteq S_A$ s.t.:

$$s, s' \in \sigma, s \neq s' \implies \text{length}(s \wedge s') \text{ is } \begin{cases} \text{even, if } A < 0 \\ \text{odd, if } A > 0 \end{cases}$$

Interleaving of words

- $A \cap B = \emptyset, u \in A^*, v \in B^*$

$$u \bullet_{A,B} v = \{w \mid w \upharpoonright A = u, w \upharpoonright B = v\}$$

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$$u \odot_{A,B} v = u \bullet_{E,F} v$$

$$a \cdot u \cdot a' \otimes_{A,B} b \cdot v \cdot b' = \{(a,b) \cdot w \cdot (a',b') \mid w \in u \odot_{A,B} v\}$$

Interpretation of MALLpol/LLpol

- Follows the pattern of Laurent's polarized games 2002
  - Orthogonal: exchanges opponant and proponent
  - Additives: disjoint unions ($S_{A \oplus B} = S_A \uplus S_B$)
  - Multiplicatives: $S_{A \otimes B} = \{s \otimes_{A,B} s' \mid s \in S_A, s \in S_B\}$
  - shifts: $S_{\downarrow A} = \{* \cdot s \cdot *' \mid s \in S_A\}$

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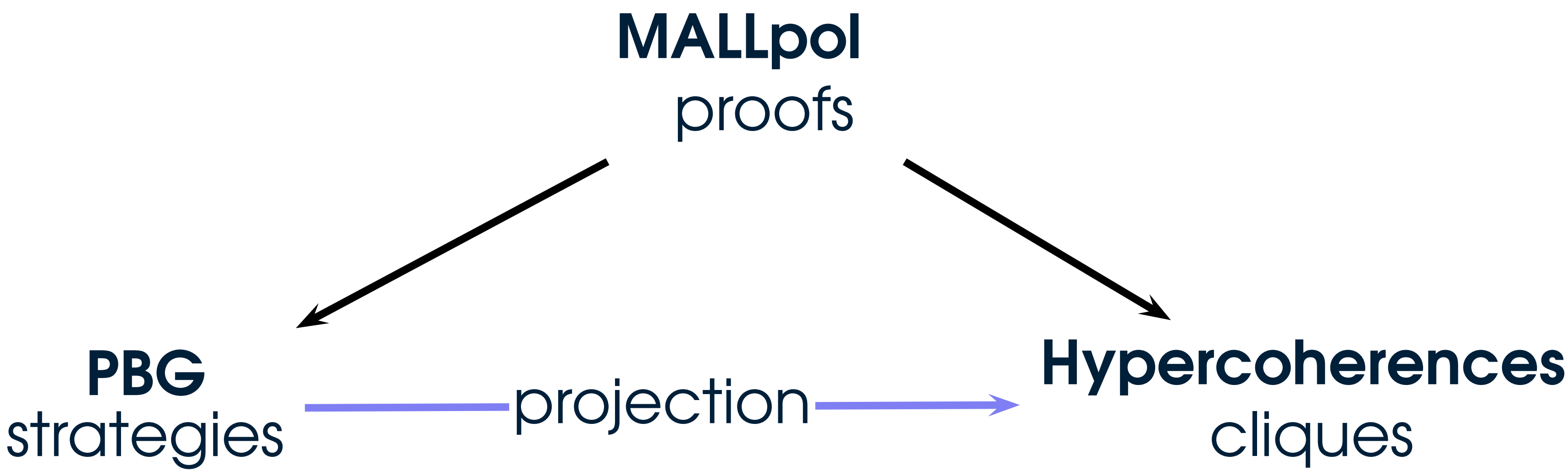
  - exponentials: $!N = \downarrow \sharp N$ (some choice for $\sharp$)

Projection



$$p([A]_{\mathsf{PBG}}) = [A]_{\mathsf{Rel}} = |[A]_{\mathsf{Hc}}|$$

$$p([\pi]_{\mathsf{PBG}}) = [\pi]_{\mathsf{Rel}} = [\pi]_{\mathsf{Hc}}$$

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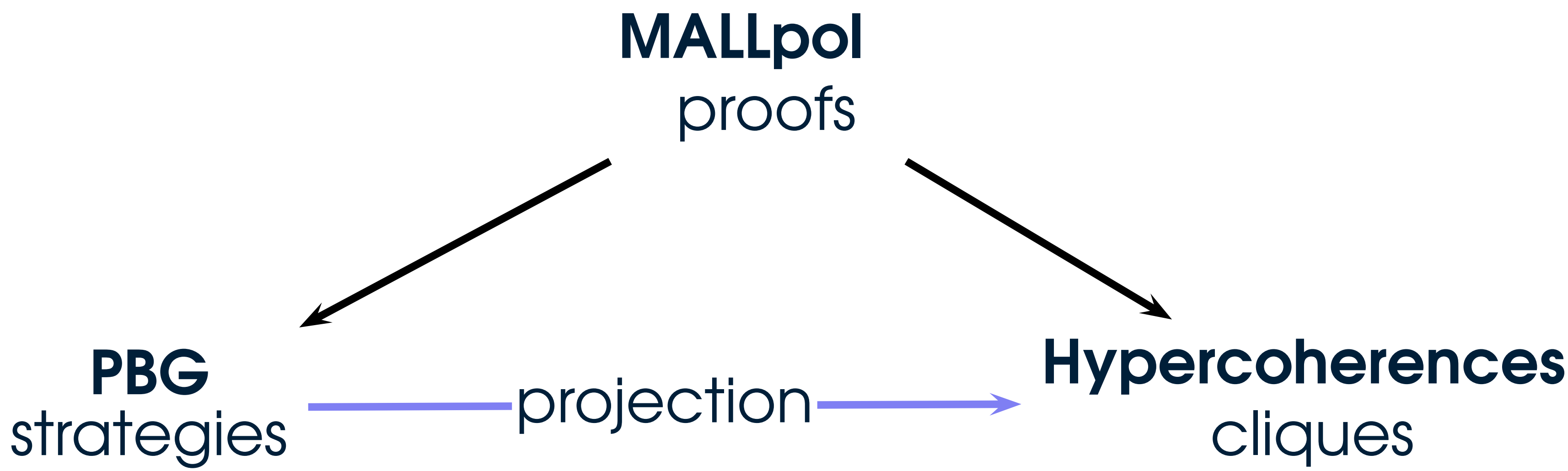

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Reversibility:

$$\mathbf{reverse}(S_A) \cong S_A$$

$$\mathbf{reverse}([\pi]_{\mathsf{PBG}}) = [\pi]_{\mathsf{PBG}}$$

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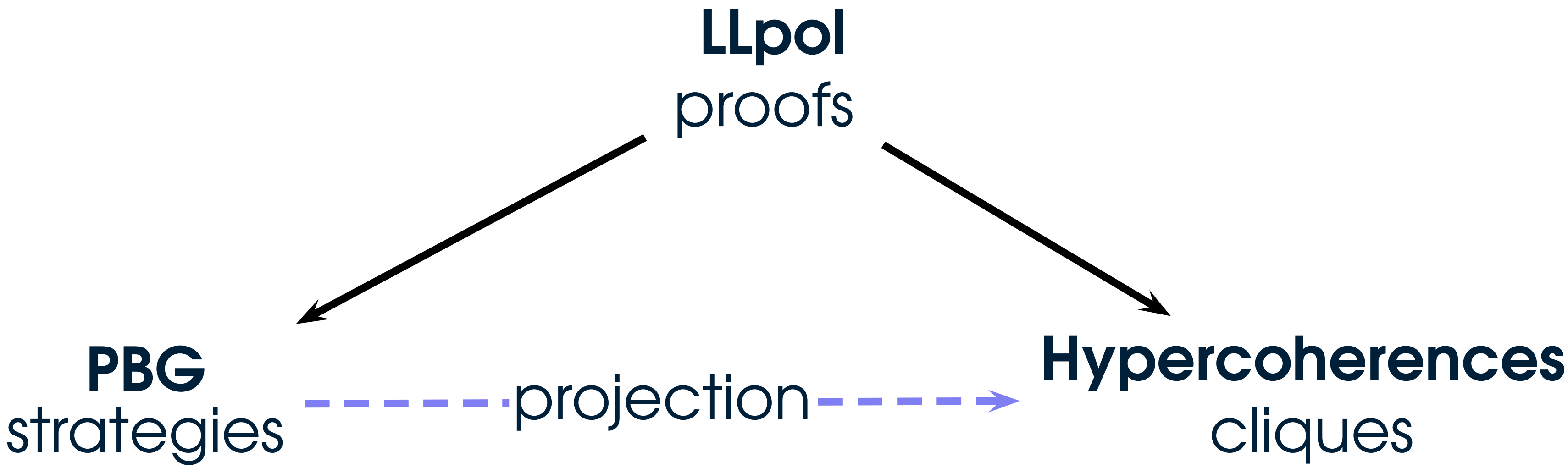

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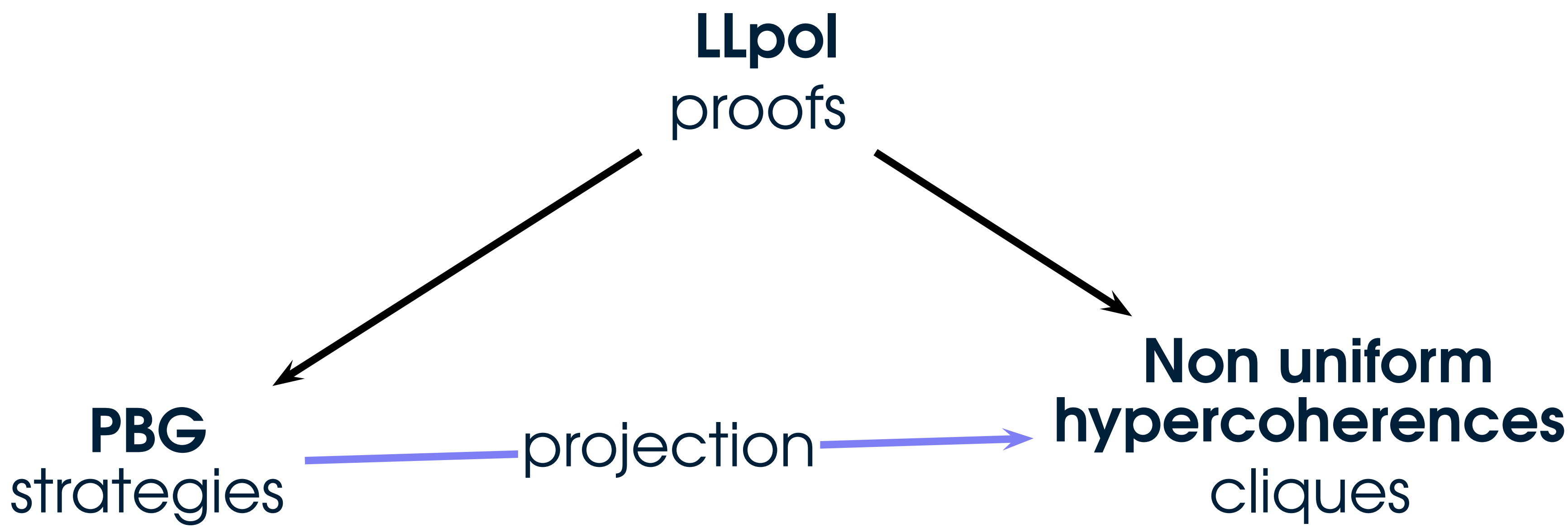

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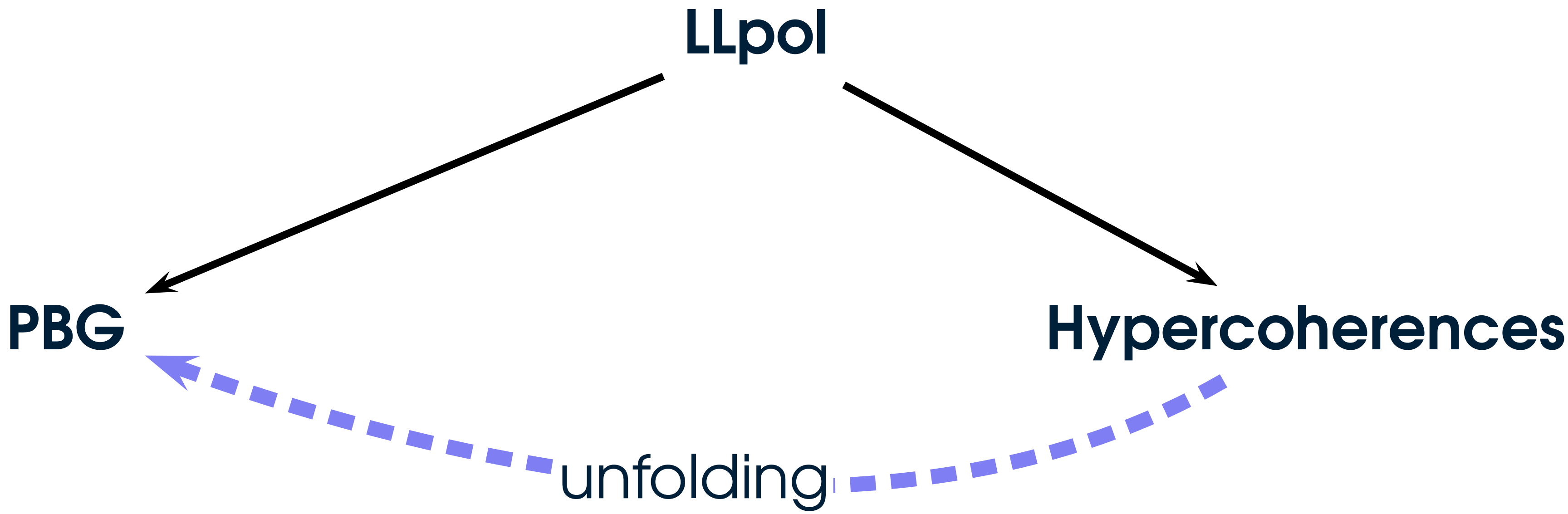
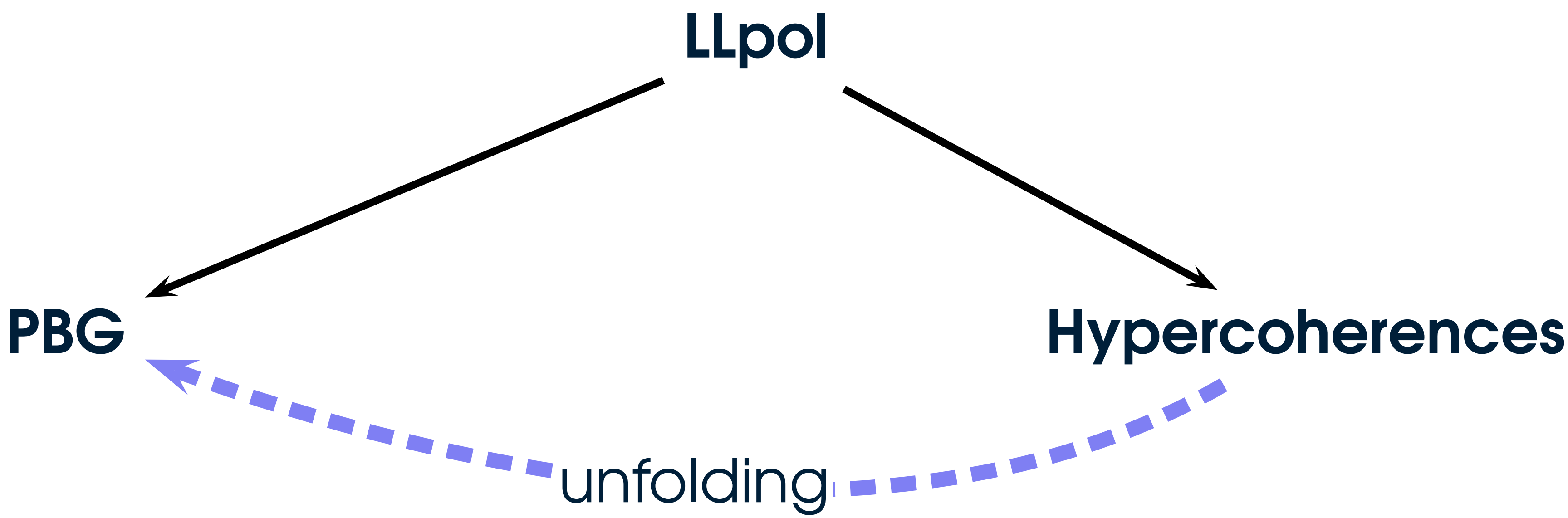
Hypercoherences unfolding



unfold(hypergraph) = tree

Hypercoherences unfolding



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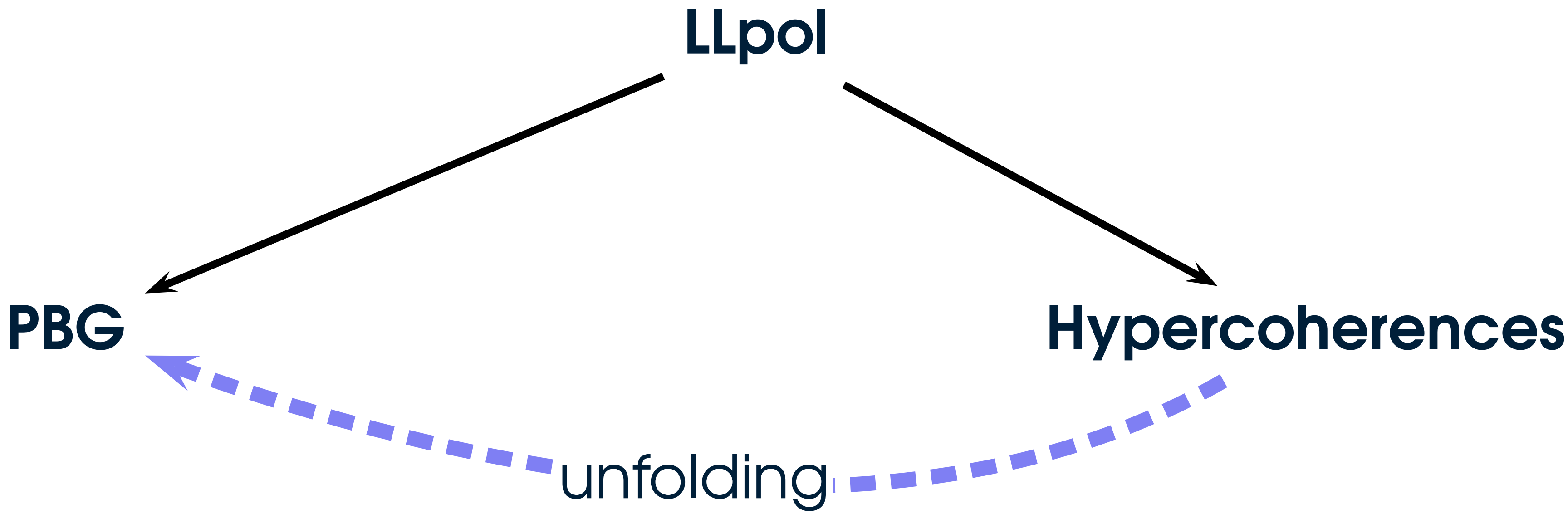

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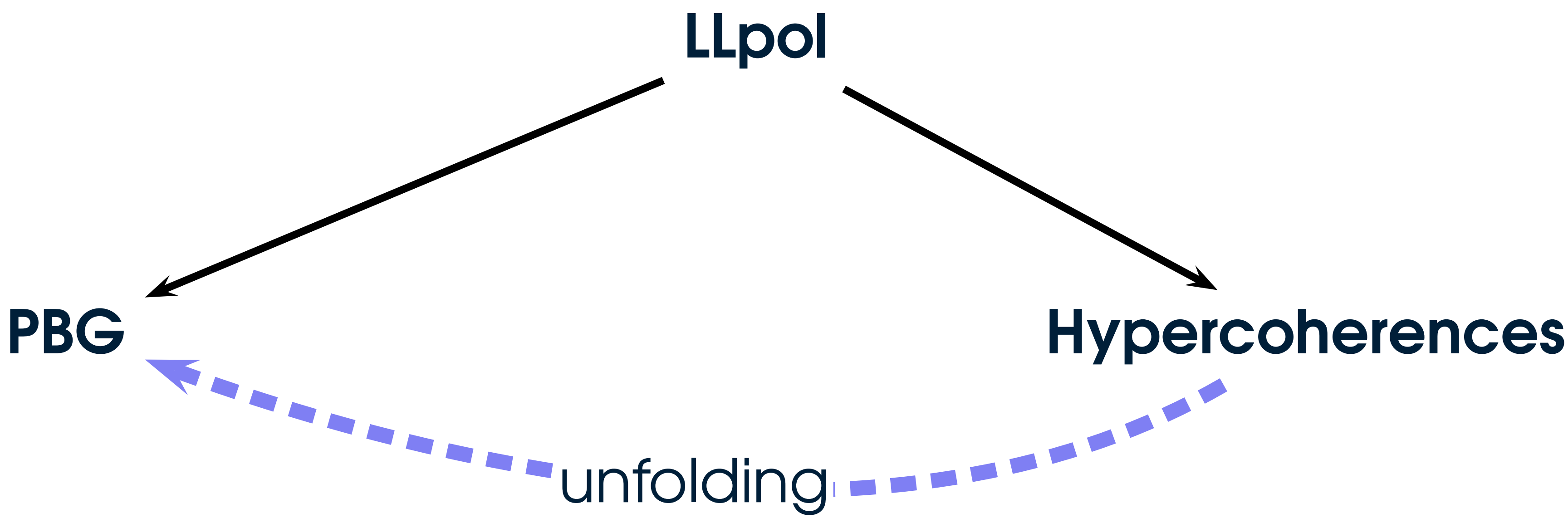

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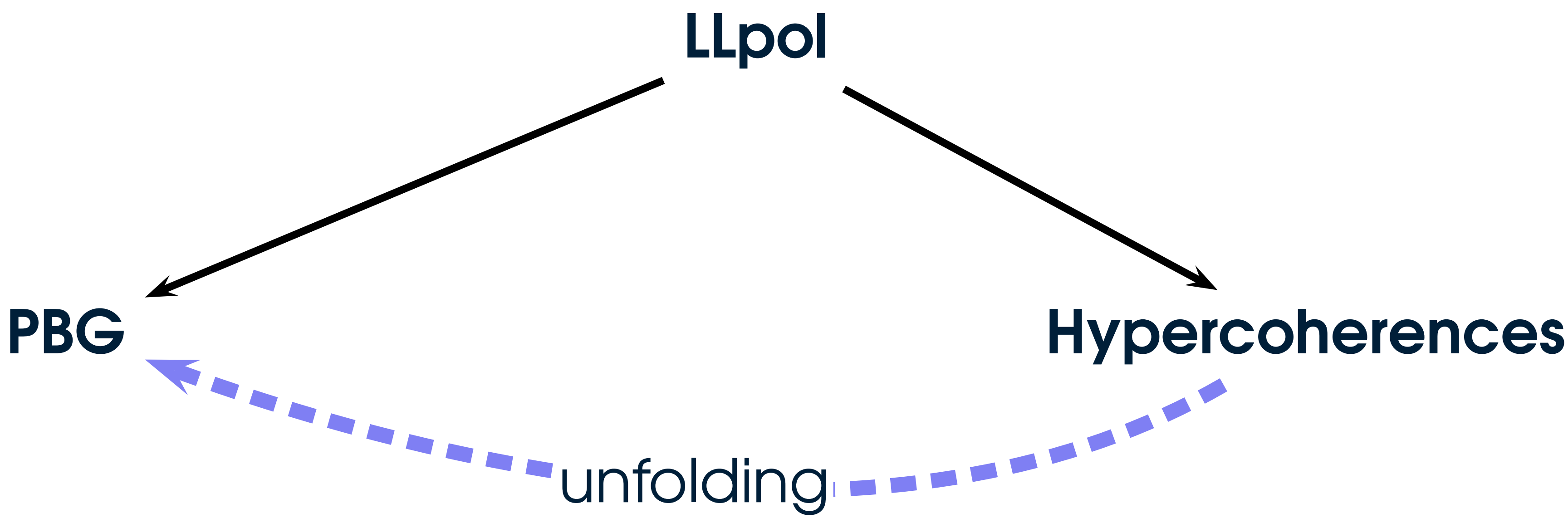

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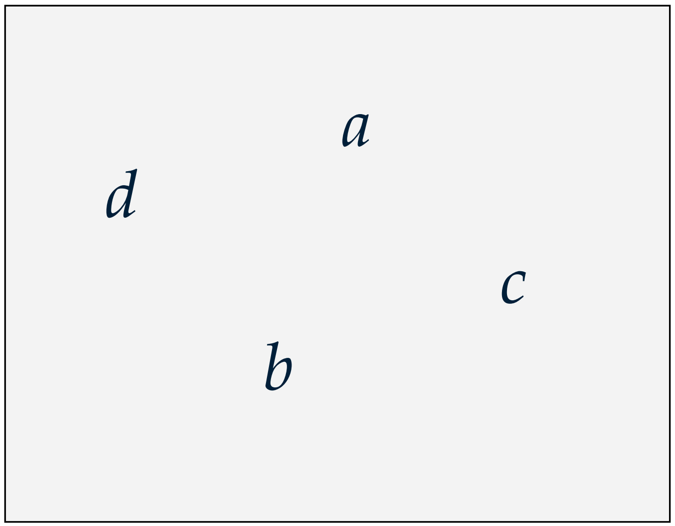
    $\mathrm{unfold}(!X) \cong \downarrow \sharp_{\mathsf{s}}\, \mathrm{unfold}(X)$
- Fails in general.

Hypercoherences unfolding

hypercoherence



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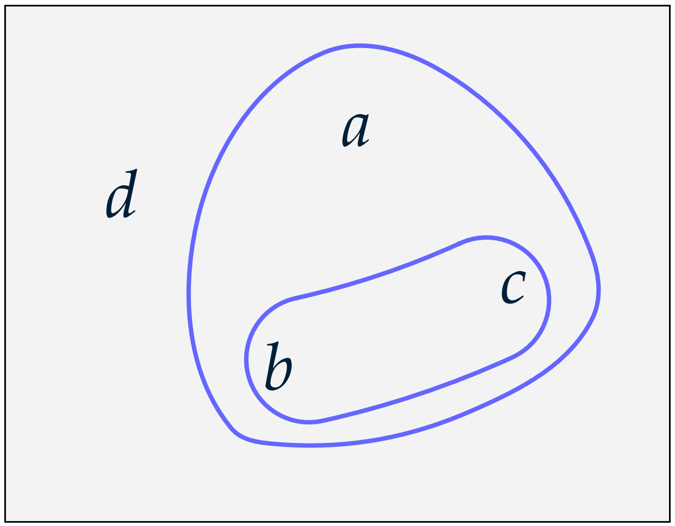

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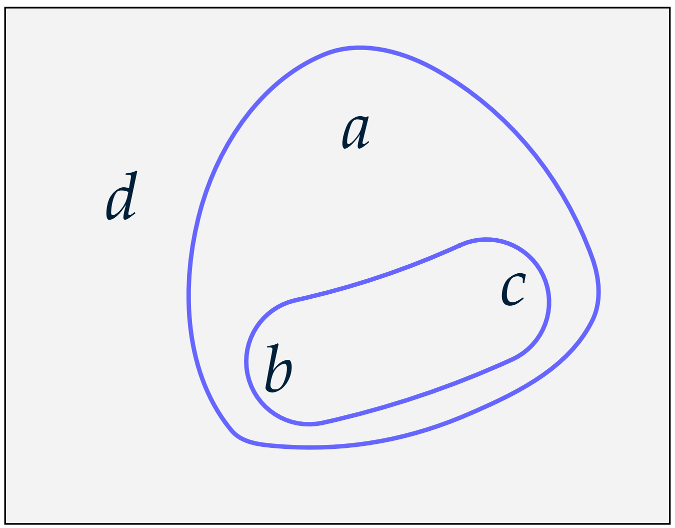

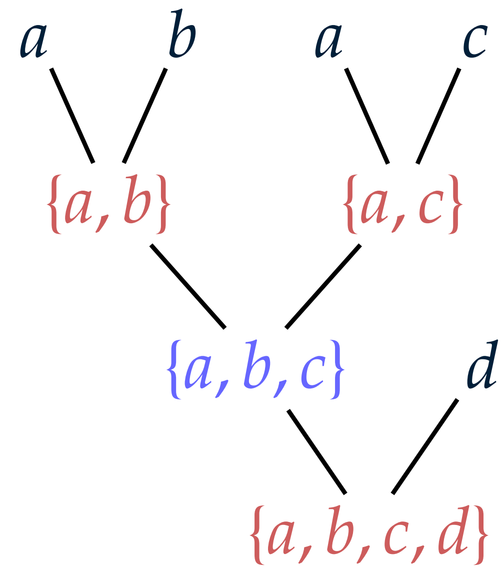

Future works

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