Logique, interactions, complexité

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1 La logique linéaire

1.1 Les formules

1.2 Les preuves (calcul des séquents)

Groupe identité

$$\frac{}{\vdash A, A^\perp}\ (ax.) \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\ \textit{(coupure)} \text{ ou } \textit{(cut)}$$

Groupe additif

$$\frac{}{\vdash \Gamma, \top}\ \textit{(top)}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \textit{(avec)} \qquad \frac{\vdash \Gamma, A_i \quad \text{où } i = 1 \text{ ou } 2}{\vdash \Gamma, A_1 \oplus A_2}\ \textit{(plus)}$$

Groupe multiplicatif

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \textit{(bot)} \qquad \frac{}{\vdash 1}\ \textit{(un)}$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \textit{(par)} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \textit{(tenseur)}$$

Groupe exponentiel

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \textit{(contraction)} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \textit{(affaiblissement)}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \textit{(déréliction)} \qquad \frac{\vdash ?A_1, \ldots, ?A_n, A}{\vdash ?A_1, \ldots, ?A_n, !A}\ \textit{(promotion)}$$

Table 1 – Les règles de la logique linéaire

Une preuve de logique linéaire est un arbre dont les arêtes sont des séquents et les nœuds sont des instances des régles (table 1.2).

1.3 Élimination des coupures

Comme en logique classique (LK), il y a un théorème d'élimination des coupures, qui établit que si le séquent $\vdash \Gamma$ est prouvable alors $\vdash \Gamma$ est prouvable sans utiliser la régle de coupure.

Remarques : un résultat d'élimination des coupures permet de montrer la consistence du système (le séquent vide n'est pas prouvable). L'élimination des coupures implique également la *propriété de la sous-formule* : toute formule apparaissant dans la preuve est une sous-formule d'une formule du séquent conclusion.

Pour éliminer les coupures on dispose d'un système de réécriture sur les preuves (donné plus loin). Ce système de réécriture utilise des régles de réduction, et des règles de commutation qui permettent de mettre les coupures en position d'être réduites.

Dans le cas la logique linéaire on a un résultat plus fort (faux dans LK) : l'élimination des coupures est fortement normalisante. Ceci signifie que quel que soit l'ordre dans lequel on élimine les coupures (on parle de stratégie de réduction), on tombe en un nombre fini d'étapes sur une preuve sans coupure.

La logique linéaire admet une syntaxe graphique, les réseaux de preuve, qui est un quotient du calcul des séquents par des équivalence telles que l'équivalence par commutation de règles. Dans les réseaux de preuve, la forme normale (le réseau de preuve sans coupure) obtenue par élimination des coupures est unique. Ceci signifie qu'à peu de choses prêts (essentiellement des commutations de règles), l'élimination des coupures est déterministe en calcul des séquents.

1.3.1 Les étapes de réduction dans l'élimination des coupures

On décrit maintenant les règles d'élimination des coupures sous la forme :

$$\boxed{\text{avant}} \xrightarrow{\beta} \boxed{\text{après}}$$

où *avant* et *après* désignent des sous-preuves de la preuve à réduire (avant, respectivement après, l'étape de réécriture).

Identité.

$$\boxed{\dfrac{(ax.)\,\dfrac{}{\vdash A, A^\perp} \qquad \dfrac{\vdots\,\pi}{\vdash \Gamma, A}}{\vdash \Gamma, A}\,(cut)} \xrightarrow{\beta} \boxed{\dfrac{\vdots\,\pi}{\vdash \Gamma, A}}$$

Additifs (plus-avec).

$$\boxed{\dfrac{(plus\ 1)\,\dfrac{\dfrac{\vdots\,\pi_1}{\vdash \Gamma, A^\perp}}{\vdash \Gamma, A^\perp \oplus B^\perp} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{\vdash \Gamma, A} \quad \dfrac{\vdots\,\pi_3}{\vdash \Gamma, B}}{\vdash \Gamma, A \,\&\, B}\,(avec)}{\vdash \Gamma, \Delta}\,(cut)} \xrightarrow{\beta} \boxed{\dfrac{\dfrac{\vdots\,\pi_1}{\vdash \Gamma, A^\perp} \quad \dfrac{\vdots\,\pi_2}{\vdash \Gamma, A}}{\vdash \Gamma, \Delta}\,(cut)}$$

Multiplicatifs (par-tenseur).

$$\boxed{\dfrac{(par)\,\dfrac{\dfrac{\vdots\,\pi_1}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \parr B} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{\vdash \Delta, A^\perp} \quad \dfrac{\vdots\,\pi_3}{\vdash \Upsilon, B^\perp}}{\vdash \Delta, \Upsilon, A^\perp \otimes B^\perp}\,(tens.)}{\vdash \Gamma, \Delta, \Upsilon}\,(cut)}$$

$$\Big\downarrow \beta$$

$$\boxed{\dfrac{\dfrac{\vdots\,\pi_1}{\vdash \Gamma, A, B} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{\vdash \Delta, A^\perp} \quad \dfrac{\vdots\,\pi_3}{\vdash \Upsilon, B^\perp}}{\vdash \Gamma, \Delta, B}\,(cut)}{\vdash \Gamma, \Delta, \Upsilon}\,(cut)}$$

Multiplicatifs (bot-un).

$$\boxed{\dfrac{(bot)\,\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma}}{\vdash \Gamma, \bot} \qquad \dfrac{}{\vdash 1}\,(un)}{\vdash \Gamma}\,(cut)} \xrightarrow{\beta} \boxed{\dfrac{\vdots\,\pi}{\vdash \Gamma}}$$

Exponentielles (affaiblissement-promotion).

$$\dfrac{(\mathit{prom.})\ \dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A}}{\vdash ?A_1,\ldots,?A_n,!A} \qquad \dfrac{\dfrac{\vdots\ \pi_2}{\vdash \Delta}}{\vdash \Delta, ?A^{\perp}}\ (\mathit{aff.})}{\vdash ?A_1,\ldots,?A_n,\Delta}\ (\mathit{cut}) \quad \xrightarrow{\ \beta\ } \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \pi_2}{\vdash \Delta}}{\vdash ?A_1,\Delta}\ (\mathit{aff.})}{\vdots}}{\vdash ?A_1,\ldots,?A_n,\Delta}\ (\mathit{aff.})$$

Exponentielles (déréliction-promotion).

$$\dfrac{(\mathit{prom.})\ \dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A}}{\vdash ?A_1,\ldots,?A_n,!A} \qquad \dfrac{\dfrac{\vdots\ \pi_2}{\vdash \Delta, A^{\perp}}}{\vdash \Delta, ?A^{\perp}}\ (\mathit{der.})}{\vdash ?A_1,\ldots,?A_n,\Delta}\ (\mathit{cut}) \quad \xrightarrow{\ \beta\ } \quad \dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A} \qquad \dfrac{\vdots\ \pi_2}{\vdash \Delta, A^{\perp}}}{\vdash ?A_1,\ldots,?A_n,\Delta}\ (\mathit{cut})$$

Exponentielles (contraction-promotion).

$$\dfrac{(\mathit{prom.})\ \dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A}}{\vdash ?A_1,\ldots,?A_n,!A} \qquad \dfrac{\dfrac{\vdots\ \pi_2}{\vdash \Delta, ?A^{\perp}, ?A^{\perp}}}{\vdash \Delta, ?A^{\perp}}\ (\mathit{cont.})}{\vdash ?A_1,\ldots,?A_n,\Delta}\ (\mathit{cut})$$

$$\Big\downarrow\ \beta$$

$$\dfrac{\dfrac{\dfrac{(\mathit{cut})\ \dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A} \qquad \dfrac{\vdots\ \pi_2}{\vdash \Delta, ?A^{\perp}, ?A^{\perp}}}{\vdash ?A_1,\ldots,?A_n,\Delta,?A^{\perp}} \qquad \dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A}}{\vdash ?A_1,\ldots,?A_n,\Delta,?A_1,\ldots,?A_n}\ (\mathit{cut})}{\vdots}\ (\mathit{cont.})}{\vdash ?A_1,\ldots,?A_n,\Delta}\ (\mathit{cont.})$$

Remarque : cette règle de réduction duplique la preuve π_1 et la coupure sans diminuer la taille de la formule coupée. On comprend mieux pourquoi établir la forte normalisation de l'élimination des coupures est difficile.

Exponentielles (promotion-promotion). Le statut de la règle suivante est plutôt d'être une règle de commutation, mais dans les réseaux elle sera en fait combinée avec les autres règles de réduction exponentielles en une seule règle. Nous la présentons donc avec les règles de réduction.

$$\dfrac{(\mathit{prom.})\ \dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A}}{\vdash ?A_1,\ldots,?A_n,!A} \qquad \dfrac{\dfrac{\vdots\ \pi_2}{\vdash ?A^{\perp},?B_1,\ldots,?B_k,B}}{\vdash ?A^{\perp},?B_1,\ldots,?B_k,!B}\ (\mathit{prom.})}{\vdash ?A_1,\ldots,?A_n,?B_1,\ldots,?B_k,!B}\ (\mathit{cut})$$

$$\Big\downarrow\ \alpha$$

$$\dfrac{\dfrac{(\mathit{prom.})\ \dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?A_1,\ldots,?A_n,A}}{\vdash ?A_1,\ldots,?A_n,!A} \qquad \dfrac{\vdots\ \pi_2}{\vdash ?A^{\perp},?B_1,\ldots,?B_k,B}}{\vdash ?A_1,\ldots,?A_n,?B_1,\ldots,?B_k,B}\ (\mathit{cut})}{\vdash ?A_1,\ldots,?A_n,?B_1,\ldots,?B_k,!B}\ (\mathit{prom.})$$

1.3.2 Règles de commutation

En calcul des séquent il y a de nombreuses règles que l'on peut faire commuter : il suffit en général que deux règles portent sur des formules différentes pour qu'elles puissent commuter (sauf contraintes particulières sur les contextes comme, en logique linéaire, avec la promotion ou le *avec*). Par exemple :

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A,B,C}\qquad\dfrac{\vdots\,\pi}{\vdash\Gamma,A,B,D}}{\vdash\Gamma,A,B,C\,\&\,D}\,(\mathit{avec})}{\vdash\Gamma,A\parr B,C\,\&\,D}\,(\mathit{par}) \xrightarrow{\alpha} \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A,B,C}}{\vdash\Gamma,A\parr B,C}\,(\mathit{par})\qquad\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A,B,D}}{\vdash\Gamma,A\parr B,C}\,(\mathit{par})}{\vdash\Gamma,A\parr B,C\,\&\,D}\,(\mathit{avec})$$

Ces règles de commutation fonctionnent en fait dans les deux sens. Mais ici nous sommes essentiellement intéressé à permettre aux coupures d'arriver en position d'être réduites. Pour cela nous fixerons l'orientation et nous ne décrirons que les commutations de la coupure avec d'autres règles. De plus nous ne présentons que les commutation avec une règle au dessus de la prémisse droite d'une coupure, mais la situation est bien entendu symétrique avec la prémisse gauche.

Commutation coupure-avec.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B}\qquad\dfrac{\vdots\,\pi''}{\vdash\Delta,A^\perp,C}}{\vdash\Delta,A^\perp,B\,\&\,C}\,(\mathit{avec})}{\vdash\Gamma,\Delta,B\,\&\,C}\,(\mathit{cut})$$

$$\Big\downarrow\,\alpha$$

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B}}{\vdash\Gamma,\Delta,B}\,(\mathit{cut})\qquad\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\vdots\,\pi''}{\vdash\Delta,A^\perp,C}}{\vdash\Gamma,\Delta,C}\,(\mathit{cut})}{\vdash\Gamma,\Delta,B\,\&\,C}\,(\mathit{avec})$$

Remarquer la duplication de la preuve π.

Commutation coupure-plus.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B_1}}{\vdash\Delta,A^\perp,B_1\oplus B_2}\,(\mathit{plus})}{\vdash\Gamma,\Delta,B_1\oplus B_2}\,(\mathit{cut}) \xrightarrow{\alpha} \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B_1}}{\vdash\Gamma,\Delta,B_1}\,(\mathit{cut})}{\vdash\Gamma,\Delta,B_1\oplus B_2}\,(\mathit{plus})$$

Commutation coupure-tenseur.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B}\qquad\dfrac{\vdots\,\pi'}{\vdash\Upsilon,C}}{\vdash\Delta,\Upsilon,A^\perp,B\otimes C}\,(\mathit{tens.})}{\vdash\Gamma,\Delta,\Upsilon,B\otimes C}\,(\mathit{cut}) \xrightarrow{\alpha} \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B}}{\vdash\Gamma,\Delta,B}\,(\mathit{cut})\qquad\dfrac{\vdots\,\pi'}{\vdash\Upsilon,C}}{\vdash\Gamma,\Delta,\Upsilon,B\otimes C}\,(\mathit{tens.})$$

Commutation coupure-par.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B,C}}{\vdash\Delta,A^\perp,B\parr C}\,(\mathit{par})}{\vdash\Gamma,\Delta,B\parr C}\,(\mathit{cut}) \xrightarrow{\alpha} \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash\Gamma,A}\qquad\dfrac{\vdots\,\pi'}{\vdash\Delta,A^\perp,B,C}}{\vdash\Gamma,\Delta,B,C}\,(\mathit{cut})}{\vdash\Gamma,\Delta,A^\perp,B,C}\,(\mathit{par})$$

Commutation coupure-bot.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp}}{\vdash \Delta, A^\perp, \bot}\,(bot)}{\vdash \Gamma, \Delta, \bot}\,(cut) \quad \xrightarrow{\alpha} \quad \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \Gamma, \Delta, \bot}\,(bot)$$

Commutation coupure-promotion. La seule commutation coupure-promotion possible est celle décrite à la fin des règles de réduction, il est nécessaire que les deux prémisses de la coupure soient des promotions.

Commutation coupure-affaiblissement.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp}}{\vdash \Delta, A^\perp, ?B}\,(\mathit{aff.})}{\vdash \Gamma, \Delta, ?B}\,(cut) \quad \xrightarrow{\alpha} \quad \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \Gamma, \Delta, ?B}\,(\mathit{aff.})$$

Commutation coupure-déréliction.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp, B}}{\vdash \Delta, A^\perp, ?B}\,(der.)}{\vdash \Gamma, \Delta, ?B}\,(cut) \quad \xrightarrow{\alpha} \quad \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp, B}}{\vdash \Gamma, \Delta, B}\,(cut)}{\vdash \Gamma, \Delta, ?B}\,(der.)$$

Commutation coupure-contraction.

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp, ?B, ?B}}{\vdash \Delta, A^\perp, ?B}\,(cont.)}{\vdash \Gamma, \Delta, ?B}\,(cut) \quad \xrightarrow{\alpha} \quad \dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, A} \qquad \dfrac{\vdots\,\pi'}{\vdash \Delta, A^\perp, ?B, ?B}}{\vdash \Gamma, \Delta, ?B, ?B}\,(cut)}{\vdash \Delta, ?B}\,(cont.)$$

Lorsqu'on a compris comment fonctionne ces règles, on les retrouve très facilement.

Nous verrons plus tard qu'il est nécessaire de faire remonter les contractions et descendre les affaiblissement par des règles de commutation de manière à éliminer ces séquences introduction (affaiblissement)/élimination (contraction). Ceci correspond à une réduction appelée réduction de Rétoré, qui n'a pas de contenu calculatoire. Il s'agit juste d'un simplification des preuves. Nous ne la verrons pas sur le calcul des séquents car il est plus simple de gérer cela dans les réseaux. Sa règle principale en calcul des séquents est la suivante :

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash \Gamma, ?A}}{\vdash \Gamma, ?A, ?A}\,(\mathit{aff.})}{\vdash \Gamma, ?A, ?A}\,(cont.) \quad \xrightarrow{\text{Rétoré}} \quad \dfrac{\vdots\,\pi}{\vdash \Gamma, ?A}$$

1.3.3 η-expansion

L'η-expansion est une réécriture sur les preuves qui en développe les axiomes sans en changer le contenu calculatoire.

Une étape de η-expansion remplace un axiome sur une formule A par une preuve (sans coupure) construite sur le connecteur principal de la formule A :

Si $A = B \mathbin{\&} C$ (ou $A = B^\perp \oplus C^\perp$) :

$$\dfrac{}{\vdash B^\perp \oplus C^\perp, B \mathbin{\&} C}\,(ax.) \quad \xrightarrow{\eta} \quad \dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax.)}{\vdash B^\perp \oplus C^\perp, B}\,(plus) \qquad \dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax.)}{\vdash B^\perp \oplus C^\perp, C}\,(plus)}{\vdash B^\perp \oplus C^\perp, B \mathbin{\&} C}\,(avec)$$

Si $A = B \parr C$ (ou $A = B^\perp \otimes C^\perp$) :

$$\dfrac{}{\vdash B^\perp \otimes C^\perp, B \parr C}\,(ax.) \xrightarrow{\eta} \dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax.) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax.)}{\vdash B^\perp C^\perp, B, C}\,(tens.)}{\vdash B^\perp C^\perp, B \parr C}\,(par)$$

Si $A = ?B$ (ou $A = !B^\perp$) :

$$\dfrac{}{\vdash ?B, !B^\perp}\,(ax.) \xrightarrow{\eta} \dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,(ax.)}{\vdash ?B, B^\perp}\,(der.)}{\vdash ?B, !B^\perp}\,(prom.)$$

L'η-expansion complète d'un axiome sur A est unique et s'obtient en un nombre fini d'étapes (le nombre de connecteurs dans A) par induction sur A.

Une preuve obtenue par η-expansion (non nécessairement complète) d'un axiome est appelée une η-expansion de l'identité. Du point de vue calculatoire, les η-expansions de l'identité laissent les preuves invariantes (à commutations de règles près dans le calcul des séquents).

Exemple de η-expansion. Si π est la preuve axiome :

$$\dfrac{}{\vdash ?X \otimes \top, !X^\perp \parr 0}\,(ax.)$$

Alors la preuve :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X, X^\perp}\,(ax.)}{\vdash ?X, X^\perp}\,(der.)}{\vdash ?X, !X^\perp}\,(prom.) \qquad \dfrac{}{\vdash \top, 0}\,(ax.)}{\vdash ?X \otimes \top, !X^\perp, 0}\,(tens.)}{\vdash ?X \otimes \top, !X^\perp \parr 0}\,(par)$$

est une η-expansion complète de π.

1.3.4 isomorphismes de type

On dit que deux formules A et B sont en isomorphes lorsqu'il existe deux preuves $\dfrac{\vdots\ \pi_1}{\vdash A^\perp, B}$ et $\dfrac{\vdots\ \pi_2}{\vdash B^\perp, A}$ telles que composer ces deux preuves π_1 et π_2 par coupure, des deux manières possibles, donne, par élimination des coupures, deux preuves $\dfrac{\vdots\ \pi_{12}}{\vdash A^\perp, A}$ et $\dfrac{\vdots\ \pi_{21}}{\vdash B^\perp, B}$ qui sont (à réécriture de Rétoré près) des η-expansions de l'identité.

Exemples d'isomorphismes de type :

$$A \otimes 1 \cong A$$
$$A \oplus 0 \cong A$$

1.3.5 Exemples d'élimination des coupures

À titre d'exemple voici deux exercices.

Exercice 1.
Montrer que $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$ est bien un isomorphisme de type.

Il nous faut deux preuves.

Une preuve π_1 de $\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), (A \otimes B) \oplus (A \otimes C)$:

$$\dfrac{\dfrac{(plus)\,\dfrac{(tens.)\,\dfrac{(ax.)\,\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{}{\vdash B^\perp, B}\,(ax.)}{\vdash A^\perp, B^\perp, A \otimes B}}{\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)} \qquad \dfrac{\dfrac{(ax.)\,\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{}{\vdash C^\perp, C}\,(ax.)}{\vdash A^\perp, C^\perp, A \otimes C}\,(tens.)}{\vdash A^\perp, C^\perp, (A \otimes B) \oplus (A \otimes C)}\,(plus)}{\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)}\,(avec)}{\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), (A \otimes B) \oplus (A \otimes C)}\,(par)$$

Une preuve π_2 de $\vdash (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp), A \otimes (B \oplus C)$.

$$\dfrac{\dfrac{\dfrac{(\textit{ax.})\,\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr B^\perp, A \otimes (B \oplus C)}\,(\textit{par}) \qquad \dfrac{\dfrac{(\textit{ax.})\,\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{\dfrac{}{\vdash C^\perp, C}\,(\textit{ax.})}{\vdash C^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, C^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr C^\perp, A \otimes (B \oplus C)}\,(\textit{par})}{\vdash (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp), A \otimes (B \oplus C)}\,(\textit{avec})$$

Coupure de π_1 et π_2 sur $(A \otimes B) \oplus (A \otimes C) = ((A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp))^\perp$ (on ne réprésente pas tout faute de place) :

$$\dfrac{\dfrac{(\textit{plus})\,\dfrac{\dfrac{\vdots\,\pi_1^1}{\vdash \ldots, A \otimes B}}{\vdash \ldots, (A \otimes B) \oplus \ldots} \qquad \dfrac{\dfrac{\vdots\,\pi_1^2}{\vdash \ldots, A \otimes C}}{\vdash \ldots, \ldots \oplus (A \otimes C)}\,(\textit{plus})}{\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)}\,(\textit{avec}) \qquad \dfrac{\vdots\,\pi_2}{\vdash (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp), A \otimes (B \oplus C)}}{\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), A \otimes (B \oplus C)}\,(\textit{cut})$$

Une étape de commutation, coupure-par :

$$\dfrac{\dfrac{\dfrac{(\textit{plus})\,\dfrac{\dfrac{\vdots\,\pi_1^1}{\vdash \ldots, A \otimes B}}{\vdash \ldots, (A \otimes B) \oplus \ldots} \qquad \dfrac{\dfrac{\vdots\,\pi_1^2}{\vdash \ldots, A \otimes C}}{\vdash \ldots, \ldots \oplus (A \otimes C)}\,(\textit{plus})}{\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)}\,(\textit{avec}) \qquad \dfrac{\vdots\,\pi_2}{\vdash \ldots}}{\vdash A^\perp, B^\perp \,\&\, C^\perp, A \otimes (B \oplus C)}\,(\textit{cut})}{\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), A \otimes (B \oplus C)}\,(\textit{par})$$

Une étape de commutation, coupure-avec (sur le avec de gauche provenant de π_1) génère deux copies de la preuve π_2 :

$$\dfrac{\dfrac{\dfrac{(\textit{plus})\,\dfrac{\dfrac{\vdots\,\pi_1^1}{\vdash \ldots, A \otimes B}}{\vdash \ldots, (A \otimes B) \oplus \ldots} \qquad \dfrac{\vdots\,\pi_2}{\vdash \ldots}}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{cut}) \qquad \dfrac{\dfrac{\dfrac{\vdots\,\pi_1^2}{\vdash \ldots, A \otimes C}}{\vdash \ldots, \ldots \oplus (A \otimes C)}\,(\textit{plus}) \qquad \dfrac{\vdots\,\pi_2}{\vdash \ldots}}{\vdash A^\perp, C^\perp, A \otimes (B \oplus C)}\,(\textit{cut})}{\vdash A^\perp, B^\perp \,\&\, C^\perp, A \otimes (B \oplus C)}\,(\textit{avec})}{\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), A \otimes (B \oplus C)}\,(\textit{par})$$

Les deux sous-preuves prémisses du dernier avec se réduisent de la même façon, nous en traitons une seule ici celle de gauche :

$$\dfrac{(\textit{plus})\,\dfrac{\dfrac{\vdots\,\pi_1^1}{\vdash \ldots, A \otimes B}}{\vdash \ldots, (A \otimes B) \oplus \ldots} \qquad \dfrac{\vdots\,\pi_2}{\vdash \ldots}}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{cut})$$

Qui est en fait :

$$\dfrac{(\textit{plus})\,\dfrac{(\textit{tens.})\,\dfrac{(\textit{ax.})\,\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash A^\perp, B^\perp, A \otimes B}}{\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)} \qquad \dfrac{\dfrac{\dfrac{(\textit{ax.})\,\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr B^\perp, A \otimes (B \oplus C)}\,(\textit{par}) \qquad \dfrac{\dfrac{(\textit{ax.})\,\dfrac{}{\vdash A^\perp, A} \qquad \dfrac{\dfrac{}{\vdash C^\perp, C}\,(\textit{ax.})}{\vdash C^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, C^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr C^\perp, A \otimes (B \oplus C)}\,(\textit{par})}{\vdash (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp), A \otimes (B \oplus C)}\,(\textit{avec})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{cut})$$

En une étape plus-avec elle devient :

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash A^\perp, B^\perp, A \otimes B}\,(\textit{tens.}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr B^\perp, A \otimes (B \oplus C)}\,(\textit{par})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{cut})$$

Puis en une étape tenseur-par et deux étapes identité (coupures contre axiomes) :

$$\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})$$

De même l'autre sous-preuve prémisse du dernier avec devient, en suivant les mêmes étapes :

$$\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash C^\perp, C}\,(\textit{ax.})}{\vdash C^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, C^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})$$

On obtient ainsi :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.}) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash C^\perp, C}\,(\textit{ax.})}{\vdash C^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, C^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp, B^\perp \,\&\, C^\perp, A \otimes (B \oplus C)}\,(\textit{avec})}{\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), A \otimes (B \oplus C)}\,(\textit{par})$$

Qui est bien une η-expansion de l'identité.

Il faut maintenant vérifier que la coupure entre π_1 et π_2 sur $A \otimes (B \oplus C) = A^\perp \parr (B^\perp \,\&\, C)$ est bien une η-expansion de l'identité :

$$\dfrac{\dfrac{\vdots\;\pi_1}{\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), (A \otimes B) \oplus (A \otimes C)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr B^\perp, A \otimes (B \oplus C)}\,(\textit{par}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash C^\perp, C}\,(\textit{ax.})}{\vdash C^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, C^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr C^\perp, A \otimes (B \oplus C)}\,(\textit{par})}{\vdash (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp), A \otimes (B \oplus C)}\,(\textit{avec})}{\vdash (A \otimes B) \oplus (A \otimes C), (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp)}\,(\textit{cut})$$

Cette fois ci, une commutation coupure-avec va générer deux copies de π_1 :

$$\dfrac{\dfrac{\dfrac{\vdots\;\pi_1}{\vdash \ldots} \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr B^\perp, A \otimes (B \oplus C)}\,(\textit{par})}{\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp}\,(\textit{cut}) \qquad \dfrac{\dfrac{\vdots\;\pi_1}{\vdash \ldots} \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash C^\perp, C}\,(\textit{ax.})}{\vdash C^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, C^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr C^\perp, A \otimes (B \oplus C)}\,(\textit{par})}{\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr C^\perp}\,(\textit{cut})}{\vdash (A \otimes B) \oplus (A \otimes C), (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp)}\,(\textit{avec})$$

Les deux sous-preuves au dessus du dernier avec sont symétriques, on en traite une seule celle de gauche :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash A^\perp, B^\perp, A \otimes B}\,(\textit{tens.})}{\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)}\,(\textit{plus}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{}{\vdash C^\perp, C}\,(\textit{ax.})}{\vdash A^\perp, C^\perp, A \otimes C}\,(\textit{tens.})}{\vdash A^\perp, C^\perp, (A \otimes B) \oplus (A \otimes C)}\,(\textit{plus})}{\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)}\,(\textit{avec})}{\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), (A \otimes B) \oplus (A \otimes C)}\,(\textit{par}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(\textit{ax.}) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(\textit{ax.})}{\vdash B^\perp, B \oplus C}\,(\textit{plus})}{\vdash A^\perp, B^\perp, A \otimes (B \oplus C)}\,(\textit{tens.})}{\vdash A^\perp \parr B^\perp, A \otimes (B \oplus C)}\,(\textit{par})}{\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp}\,(\textit{cut})$$

Il y a une étape de commutation coupure-par :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\LeftLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes B$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, C^\perp, A \otimes C$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\RightLabel{(par)}
\UnaryInfC{$\vdash A^\perp \parr (B^\perp \,\&\, C^\perp), (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash B^\perp, B \oplus C$}
\RightLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes (B \oplus C)$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp, B^\perp$}
\RightLabel{(par)}
\UnaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp$}
\end{prooftree}
$$

Puis une étape de réduction par-tenseur :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\LeftLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes B$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, C^\perp, A \otimes C$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash B^\perp, B \oplus C$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp, B^\perp$}
\RightLabel{(par)}
\UnaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp$}
\end{prooftree}
$$

Une étape coupure identité :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\LeftLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes B$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, C^\perp, A \otimes C$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash A^\perp, B^\perp \,\&\, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash B^\perp, B \oplus C$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp, B^\perp$}
\RightLabel{(par)}
\UnaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp$}
\end{prooftree}
$$

Une étape coupure avec-plus :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\LeftLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes B$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp, B^\perp$}
\RightLabel{(par)}
\UnaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp$}
\end{prooftree}
$$

Et enfin une coupure identité :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\LeftLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes B$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)$}
\RightLabel{(par)}
\UnaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp$}
\end{prooftree}
$$

Ce qui donne pour la preuve globale :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\LeftLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes B$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, B^\perp, (A \otimes B) \oplus (A \otimes C)$}
\LeftLabel{(par)}
\UnaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr B^\perp$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{(tens.)}
\BinaryInfC{$\vdash A^\perp, C^\perp, A \otimes C$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp, C^\perp, (A \otimes B) \oplus (A \otimes C)$}
\RightLabel{(par)}
\UnaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), A^\perp \parr C^\perp$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash (A \otimes B) \oplus (A \otimes C), (A^\perp \parr B^\perp) \,\&\, (A^\perp \parr C^\perp)$}
\end{prooftree}
$$

Qui est bien une η-expansion de l'indentité.

Remarque : en relisant les preuves précédentes nous pouvons remarquer que nous avons également démontré que $A \parr (B \,\&\, C) \cong (A \parr B) \,\&\, (A \parr B)$. Plus généralement un isomorphisme de type passe toujours à l'orthogonal.

Exercice 2.
Montrer que $!(A \,\&\, B) \cong\, !A \otimes !B$ *est bien un isomorphisme de type.*

Preuve π_1 de $!(A \,\&\, B) \vdash\, !A \otimes !B$:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\RightLabel{$(cont.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\end{prooftree}
$$

Preuve π_2 de $!A \otimes !B \vdash !(A \mathbin{\&} B)$:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \mathbin{\&} B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \mathbin{\&} B)$}
\RightLabel{(par)}\UnaryInfC{$\vdash ?A^\perp \parr ?B^\perp, !(A \mathbin{\&} B)$}
\end{prooftree}
$$

Coupure de π_1 et π_2 sur $!A \otimes !B = (?A^\perp \parr ?B^\perp)^\perp$:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\RightLabel{$(cont.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \mathbin{\&} B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \mathbin{\&} B)$}
\RightLabel{(par)}\UnaryInfC{$\vdash ?A^\perp \parr ?B^\perp, !(A \mathbin{\&} B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \mathbin{\&} B)$}
\end{prooftree}
$$

Après commutation avec la contraction de π_1 :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \mathbin{\&} B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \mathbin{\&} B)$}
\RightLabel{(par)}\UnaryInfC{$\vdash ?A^\perp \parr ?B^\perp, !(A \mathbin{\&} B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \mathbin{\&} B)$}
\RightLabel{$(cont.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \mathbin{\&} B)$}
\end{prooftree}
$$

On élimine la coupure tenseur-par :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\AxiomC{}
\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \mathbin{\&} B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \mathbin{\&} B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), !(A \mathbin{\&} B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \mathbin{\&} B)$}
\RightLabel{$(cont.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \mathbin{\&} B)$}
\end{prooftree}
$$

Coupure promotion-promotion (on omet de représenter certaines parties inchangées de la preuve) :

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{(\textit{avec})}
\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \& B$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A \& B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), !(A \& B)$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \& B)$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \& B)$}
\end{prooftree}
$$

Commutation coupure et règle avec :

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\RightLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{avec})}
\BinaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A \& B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), !(A \& B)$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \& B)$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \& B)$}
\end{prooftree}
$$

Deux étapes, coupure promotion-affaiblissement et commutation cut-affaiblissement :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{avec})}
\BinaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A \& B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), !(A \& B)$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \& B)$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \& B)$}
\end{prooftree}
$$

Deux étapes, coupure promotion-déreliction et promotion-promotion :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{avec})}
\BinaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A \& B$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A \& B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \& B)$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \& B)$}
\end{prooftree}
$$

Deux étapes coupure identité (axiome), et commutation cut-avec :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?A^\perp, A$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(cut)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?A^\perp, ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A \,\&\, B$}
\RightLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\RightLabel{(cont.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\end{prooftree}
$$

Deux étapes, commutation promotion-affaiblissement et coupure promotion-affaiblissement :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(cut)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A \,\&\, B$}
\RightLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\RightLabel{(cont.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\end{prooftree}
$$

Une étape, coupure promotion-déréliction :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(cut)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A \,\&\, B$}
\RightLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\RightLabel{(cont.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\end{prooftree}
$$

une étape, coupure identité (axiome) :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(aff.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A \,\&\, B$}
\RightLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\RightLabel{(cont.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !(A \,\&\, B)$}
\end{prooftree}
$$

On tombe sur une preuve qui n'est pas exactement la η-expansion d'une identité. Pour retomber sur nos pates il faut utiliser l'équivalence de Rétoré qui établit que supprimer une séquence affaiblissement-contraction (introduction d'un second $?A$ et contraction avec un $?A$ déjà présent) ne change pas le comportement calculatoire de la preuve. Ici on peut faire un peu de réduction de Rétoré en faisant au préalable une commutation affaiblissement-avec, puis une commutation affaiblissement-promotion.

Coupure de π_1 et π_2 sur $!(A \,\&\, B) = (?(A^\perp \oplus B^\perp))^\perp$:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(plus)}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(prom.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{(tens.)}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\RightLabel{(cont.)}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\AxiomC{}
\LeftLabel{(ax.)}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(der.)}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(aff.)}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{(ax.)}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(der.)}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{(aff.)}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{(avec)}
\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{(prom.)}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(par)}
\UnaryInfC{$\vdash ?A^\perp \parr ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Commutation avec par :

$$
\begin{prooftree}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\RightLabel{$(cont.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(par)}\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Coupure contraction-promotion (duplique le reste de π_2, on ne représente pas tout) :

$$
\begin{prooftree}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B$}
\AxiomC{$\vdots\ \pi_2'$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B, ?A^\perp, ?B^\perp$}
\AxiomC{$\vdots\ \pi_2'$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(par)}\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Traitons la sous-preuve au dessus de la coupure la plus haute. Il faut commencer par une commutation de la coupure avec le tenseur (nota bene : on considère que la coupure a porté sur le $?(A^\perp \oplus B^\perp)$ le plus à gauche). Voici ce que devient cette sous-preuve après l'étape de commutation tenseur-coupure :

$$
\begin{prooftree}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A$}
\AxiomC{$\vdots\ \pi_2'$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B, ?A^\perp, ?B^\perp$}
\noLine\UnaryInfC{$\vdots$}
\end{prooftree}
$$

Une étape de commutation promotion (on a développé π_2') :

$$
\begin{prooftree}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), A$}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(cut)}\BinaryInfC{$\vdash A, ?A^\perp, ?B^\perp$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B, ?A^\perp, ?B^\perp$}
\noLine\UnaryInfC{$\vdots$}
\end{prooftree}
$$

Une étape de réduction déreliction-promotion :

$$
\begin{prooftree}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, A$}
\AxiomC{}\LeftLabel{$(ax.)$}\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{(cut)}\BinaryInfC{$\vdash A, ?A^\perp, ?B^\perp$}
\LeftLabel{$(prom.)$}\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}\RightLabel{$(ax.)$}\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{$(der.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{$(prom.)$}\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{$(tens.)$}\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B, ?A^\perp, ?B^\perp$}
\noLine\UnaryInfC{$\vdots$}
\end{prooftree}
$$

Et deux étapes de réduction, avec-plus puis identité :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\LeftLabel{(\textit{prom.})}
\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{(\textit{tens.})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B, ?A^\perp, ?B^\perp$}
\UnaryInfC{$\vdots$}
\end{prooftree}
$$

En revenant à la preuve globale, il reste une coupure à éliminer :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\LeftLabel{(\textit{prom.})}
\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\RightLabel{(\textit{tens.})}
\BinaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B, ?A^\perp, ?B^\perp$}
\AxiomC{$\vdots\ \pi_2'$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?B^\perp$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{par})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

On effectue une étape de commutation avec le tenseur (et on développe π_2') :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\LeftLabel{(\textit{prom.})}
\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), !B$}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{(\textit{avec})}
\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash !B, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{tens.})}
\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?B^\perp$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{par})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Une étape de commutation avec une promotion :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\LeftLabel{(\textit{prom.})}
\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{plus})}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?(A^\perp \oplus B^\perp), B$}
\AxiomC{}
\LeftLabel{(\textit{ax.})}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{(\textit{ax.})}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{(\textit{der.})}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{(\textit{aff.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{(\textit{avec})}
\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, !(A \,\&\, B)$}
\RightLabel{(\textit{cut})}
\BinaryInfC{$\vdash B, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{prom.})}
\UnaryInfC{$\vdash !B, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{tens.})}
\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?B^\perp$}
\RightLabel{(\textit{cont.})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(\textit{par})}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Une étape de réduction dérélection-promotion :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\LeftLabel{$(prom.)$}
\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}
\RightLabel{$(ax.)$}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(plus)$}
\UnaryInfC{$\vdash A^\perp \oplus B^\perp, B$}
\AxiomC{}
\LeftLabel{$(ax.)$}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\AxiomC{}
\RightLabel{$(ax.)$}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(avec)$}
\BinaryInfC{$\vdash ?A^\perp, ?B^\perp, A \,\&\, B$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash B, ?A^\perp, ?B^\perp$}
\RightLabel{$(prom.)$}
\UnaryInfC{$\vdash !B, ?A^\perp, ?B^\perp$}
\RightLabel{$(tens.)$}
\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(par)}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Puis une étape de réduction plus-avec :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\LeftLabel{$(prom.)$}
\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}
\RightLabel{$(ax.)$}
\UnaryInfC{$\vdash B^\perp, B$}
\AxiomC{}
\RightLabel{$(ax.)$}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash B, ?A^\perp, ?B^\perp$}
\RightLabel{$(prom.)$}
\UnaryInfC{$\vdash !B, ?A^\perp, ?B^\perp$}
\RightLabel{$(tens.)$}
\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(par)}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Et enfin une réduction identité :

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$(ax.)$}
\UnaryInfC{$\vdash A^\perp, A$}
\LeftLabel{$(der.)$}
\UnaryInfC{$\vdash ?A^\perp, A$}
\LeftLabel{$(aff.)$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, A$}
\LeftLabel{$(prom.)$}
\UnaryInfC{$\vdash !A, ?A^\perp, ?B^\perp$}
\AxiomC{}
\RightLabel{$(ax.)$}
\UnaryInfC{$\vdash B^\perp, B$}
\RightLabel{$(der.)$}
\UnaryInfC{$\vdash ?B^\perp, B$}
\RightLabel{$(aff.)$}
\UnaryInfC{$\vdash ?A^\perp, ?B^\perp, B$}
\RightLabel{$(prom.)$}
\UnaryInfC{$\vdash !B, ?A^\perp, ?B^\perp$}
\RightLabel{$(tens.)$}
\BinaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?B^\perp$}
\RightLabel{$(cont.)$}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp, ?B^\perp$}
\RightLabel{(par)}
\UnaryInfC{$\vdash !A \otimes !B, ?A^\perp \parr ?B^\perp$}
\end{prooftree}
$$

Comme pour la preuve précédente, il faut effectuer un peu de réduction de Rétoré , faire descendre les affaiblissements par commutation et les annuler avec les contractions.

Ces exemples montrent que l'élimination des coupures (plus réduction de Rétoré) n'est pas chose aisée en calcul des séquents.

Heureusement la logique linéaire possède une syntaxe graphique, les réseaux de preuve, qui nous évitera toutes les étapes de commutation. Dans cette syntaxe, nous nous concentrerons sur le fragment multiplicatif sans constantes, puis multiplicatif-exponentiel sans constantes. Il existe différents moyens plus ou moins élégants de gérer les additifs et les constantes dans cette syntaxe, mais c'est au prix de nombreuses complications que nous éviterons ici.