Projecting games on hypercoherences

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Abstract. We compare two interpretations of programming languages: game semantics (a dynamic semantics dealing with computational traces) and hypercoherences (a static semantics dealing with results of computation). We consider polarized bordered games which are Laurent's polarized games endowed with a notion of terminated computation (the border) allowing for a projection on hypercoherences. The main result is that the projection commutes to the interpretation of linear terms (exponential-free proofs of polarized linear logic). We discuss the extension to general terms.

The Curry-Howard isomorphism establishes a correspondence between proofs and programs and between formulæ and types. In this paper we adopt the logical point of view on computation and we use the sequent calculus syntax (where cut-elimination represents dynamic of computation) of Girard's linear logic [10] (LL for short). Let us recall that LL splits logic into a linear fragment, where resources are consumed when used, and an exponential fragment, allowing data copying and erasing through structural rules.

1 Introduction

In denotational semantics, an agent (a program, a term or a proof) is represented as a structure describing all its possible interactions with other agents. Static semantics (e.g. hypercoherences [7]) focus on results of interactions while dynamic semantics (e.g. game semantics) focus on interaction histories (computational traces called *plays* in game semantics). This difference is somewhat the same as the difference between a function (static) and an algorithm (dynamic).

In the fifties, Kreisel introduced partial equivalence relations (PER) to deal with higher order functions presented in an operational manner (algorithms, recursive functions, proofs). Partiality of the equivalence relation comes from the fact that a higher order algorithm can separate two algorithms which compute the same function whereas a higher order function cannot.

The hypercoherence semantics of LL [7] has been introduced to give an "extensional" account of sequentiality [6]. Hypercoherences, just as coherence spaces, are built by adding a graph structure to the objets of the *relational* *model*. In this model, formulæ are interpreted as sets of points (results of computation) and agents as relations between them.

In [8], Ehrhard shows that hypercoherences form the *extensional collapse* (the quotient by PERs) of sequential algorithms (a model of PCF introduced by Berry and Curien in [3] and which has been shown by Lamarche to be a game model [13]). This result has been proved again by Longley and Van Oosten, with different methods (see [16, 20]) and has been independently extended to other game semantics by Laird [12] and Melliès [18]. This relates surprisingly games to hypercoherences in the simple types hierarchy and shows that hypercoherences carry an implicit representation of the dynamic of computation. Our goal is to make the dynamical content of hypercoherences more explicit, in LL and not only in the simple types hierarchy. The various proofs of Ehrhard's result we already mentioned do not give a clear understanding of this dynamical content.

In [17], P.-A. Melliès gives a new proof of Ehrhard's result which clarifies the relation between games and hypercoherences, for simple types.

In [2], Baillot, Danos, Ehrhard and Regnier present the projection of a standard game model of multiplicative exponential linear logic onto a suitable static model based on the relational semantics by means of a lax time forgetful functor. Since this functor is lax, the projection of the game interpretation of a proof is included into its static interpretation, but not the converse, in general.

Our approach to the comparaison of games and hypercoherences consists in finding a suitable framework with a projection of plays onto points of the relational model, and then working out, on top of this framework, a precise relation between the hypergraph structure of hypercoherences and the dynamical structure of games.

In section 2, inspired by the *rigid parallel unfolding* of hypercoherences of [9], we introduce *polarized bordered games* (PBG for short). PBGs are polarized games (a game model of both polarized linear logic and linear logic with polarities [15, 14]) endowed with a *border* which is a set of plays to be considered as the *terminated* plays.

We present the PBG interpretation of the linear fragment (MALLpol for short) of linear logic with polarities (LLpol for short).

The terminated plays of a PBG are the plays which can be projected onto the points of the relational model. Thanks to this additional structure, the projection commutes to the interpretation of proofs of MALLpol. But, in general, the projection of a strategy (other than the interpretation of a proof) is not a clique in hypercoherences.

A peculiar reversibility property of PBGs is also presented.

We next show, in section 3, how to extend the PBG semantics to LLpol and ILL, considering two interpretations of the exponentials. One is a version "with a border" of the exponential of Berry-Curien's sequential algorithms and the other is a new kind of exponential.

In section 4, we try to relate the hypergraph structure of hypercoherences with PBGs. We briefly present an unfolding of hypercoherences into *tower trees*, generalizing a construction given in [9] with the aim of disclosing the dynamical content of hypercoherences. This unfolding maps the hypercoherence interpretation of additive and multipicative connectives to their PBG interpretation. This is not the case, in general, for exponentials.

We end this section by recalling the notion of hypercoherence and the syntax and the rules of linear logic with polarities.

A hypercoherence X is just a hypergraph consisting of a countable set of vertices $|X|$, the *web*, together with a set of hyperedges Γ, the *coherence*. More precisely, Γ is a subset of $\mathcal{P}^*_{\text{fin}}(|X|)$, the set of non-empty finite subsets of the web. In hypercoherences, each singleton is coherent. The *strict coherence* Γ^* is just coherence without singletons, *i.e.* $\Gamma \setminus \{\{a\} \mid a \in |X|\}$. The *incoherence* $\Gamma^\perp$ is the complementary set of Γ^* in $\mathcal{P}^*_{\text{fin}}(|X|)$. A clique is a subset x of the web such that $\mathcal{P}^*_{\text{fin}}(x) \subseteq \Gamma$.

The *orthogonal* is interpreted by the exchange of coherence and incoherence.

The interpretation of LL in hypercoherences follows the pattern of its interpretation in coherence spaces (see [7]).

Linear logic with polarities, LLpol, is the fragment of LL restricted to formulæ:

$$P := 0 \mid 1 \mid P \oplus P \mid P \otimes P \mid !N \mid \alpha^\perp$$
$$N := \top \mid \perp \mid N \mathbin{\&} N \mid N \parr N \mid ?P \mid \alpha$$

where α denotes atoms, P stands for positive formulæ and N for negative formulæ. In LLpol sequents contains at most one positive formula. We use Γ to range over contexts containing at most one positive formula and $\mathcal{N}, \mathcal{N}'$ to range over contexts made of negative formulæ only. The rules are just the ordinary rules of LL:

$$\frac{}{\vdash \alpha, \alpha^\perp}\,(ax.) \qquad \frac{}{\vdash 1}\,(one) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\,(bot) \qquad \frac{}{\vdash \top, \Gamma}\,(top)$$

$$\frac{\vdash \mathcal{N}, P_i \quad (i = 1, 2)}{\vdash \mathcal{N}, P_1 \oplus P_2}\,(plus) \qquad \frac{\vdash \Gamma, N \qquad \vdash \Gamma, N'}{\vdash \Gamma, N \mathbin{\&} N'}\,(with)$$

$$\frac{\vdash N, N', \Gamma}{\vdash N \parr N', \Gamma}\,(par) \qquad \frac{\vdash \mathcal{N}, P \qquad \vdash \mathcal{N}', P'}{\vdash \mathcal{N}, \mathcal{N}', P \otimes P'}\,(tens.)$$

$$\frac{\vdash ?P_1, \ldots, ?P_n, N}{\vdash ?P_1, \ldots, ?P_n, !N}\,(prom.) \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, ?P}\,(der.) \qquad \frac{\vdash \mathcal{N}, N^\perp \qquad \vdash N, \Gamma}{\vdash \mathcal{N}, \Gamma}\,(cut)$$

$$\frac{\vdash \Gamma}{\vdash ?P, \Gamma}\,(weak.) \qquad \frac{\vdash ?P, ?P, \Gamma}{\vdash ?P, \Gamma}\,(cont.)$$

The linear subsystem of LLpol, denoted by MALLpol, is LLpol without the structural rules (weakening and contraction). In MALLpol we denote ! by $\downarrow$ and ? by $\uparrow$ and these modalities are called shifts, since they are not real exponentials but shifts of polarities.

We use the notation [ ] for multisets while the notation { } is, as usual, for sets.

2 Polarized bordered games

If A is an alphabet (a set) then A^* denotes the set of words on A (the finite sequences, up to reindexing). We denote by ε the empty word and by $w \cdot m$ the concatenation of the words w and m. The longest common prefix of two words w and w' is denoted by $w \wedge w'$.

If A and B are disjoint sets and if $C \subseteq A \cup B \cup (A \times B) \cup (B \times A)$ then *restriction to* A is the function from C^* to A^* defined by $\varepsilon \upharpoonright A = \varepsilon$, $w \cdot l \upharpoonright A = (w \upharpoonright A) \cdot a$ if $l = a \in A$ or if $l = (a, b) \in A \times B$ or if $l = (b, a) \in A \times B$ and $w \cdot l \upharpoonright A = w \upharpoonright A$ if $l \in B$.

A set of words E is seen as a *forest*: vertices are the non empty prefixes of the elements of E, roots are the one letter words and an edge relates two words when one of them is the immediate prefix of the other. Conversely, a forest can always be described as a set of words. This set is usually taken prefix-closed for ensuring some unicity of the representation, but this restriction is not convenient in the present setting.

Here, a *tree isomorphism* between $E \subseteq A^*$ and $F \subseteq B^*$ is a bijection $f : E \to F$ such that for each pair $s, s' \in E$, the length of $f(s) \wedge f(s')$ is equal to the length of $s \wedge s'$. The usual notion of tree of forest isomophism would normally corresponds to a standard representation of trees as prefix-closed sets of words. Here, E and F may have isomorphic prefix closures without being isomorphic.

In what follows, logical polarities and game polarities can be identified. So, negative is opponent and positive is player.

Definition 1 (polarized bordered game). *A PBG A is a tuple (ϵ, A^-, A^+, S) where $\epsilon \in \{-, +\}$ is the polarity of A ($\overline{\epsilon}$ denotes the opposite polarity), A^- and A^+ are two countable and disjoint sets, and where S, the* border *of A, is a subset of $(A^\epsilon \cdot A^{\overline{\epsilon}})^*$. The elements of the prefix closure of S, denoted as P_A, are the* plays *of A, the elements of S are the* terminated plays *of A, and $P_A^{\overline{\epsilon}}$ (resp. P_A^{ϵ}) denotes the even prefix (resp. odd prefix) closure of S.*

If S is a set of words then a subset x of S is *even-deterministic* (resp. *odd-deterministic*) if for each two elements s and s' of x which are incomparable for the prefix order, the length of $s \wedge s'$ is even (resp. odd).

Definition 2 (strategies). *In a PBG A, a* strategy *is a deterministic subset of S_A: even-deterministic if $\epsilon_A = -$, or odd-deterministic if $\epsilon_A = +$.*

Let $E \subseteq C^*$ and $E' \subseteq C'^*$ be two sets of words on two disjoint alphabets. Let $m \in E$ and $m' \in E'$. The set $m \bullet_{C,C'} m'$ is the subset of words w on the alphabet $C \cup C'$ such that $w \upharpoonright C = m$ and $w \upharpoonright C' = m'$. And the set $E \bullet_{C,C'} E'$ is equal to

$$E \bullet_{C,C'} E' = \bigcup_{\substack{m \in E \\ m' \in E}} m \bullet_{C,C'} m'. \tag{1}$$

If $C = A \cdot (B \cdot B)^* \cdot A$ and $C' = A' \cdot (B' \cdot B')^* \cdot A'$, where A, A', B, B' are pairwise disjoint, then we also denote by $\odot_{A,A'}$ the operation $\bullet_{C,C'}$.

On words of even length, the operation $\otimes_{A,A'}$ is defined as follows:

$$d_1 \cdot m \cdot d_2 \otimes_{A,A'} d'_1 \cdot m' \cdot d'_2 = \{(d_1, d'_1) \cdot w \cdot (d_2, d'_2) \mid w \in m \odot_{A,A'} m'\} \tag{2}$$

where $d_1, d_2 \in A \cup B$ and $d'_1, d'_2 \in A' \cup B'$, and if one of the two words t or t' is empty then $t \otimes t' = \emptyset$.

We define the logical connectives on PBGs respecting the LLpol polarity restrictions.

The *orthogonal* of $A = (\epsilon_A, A^-, A^+, S_A)$ is the PBG $A^\perp = (\overline{\epsilon_A}, A^-, A^+, S_A)$. *Top* is the PBG $\top = (-, \emptyset, \emptyset, \emptyset)$. *Bot* is the PBG $\perp = (-, \{*\}, \{*'\}, \{**'\})$. Let $A = (-, A^-, A^+, S_A)$ and $B = (-, B^-, B^+, S_B)$ be two negative disjoint PBGs (when A and B are not disjoint we separate them by using subscripts). The *positive shift* of A is the PBG $\downarrow A = (+, A^- \cup \{*'\}, A^+ \cup \{*\}, \{*\} \cdot S_A \cdot \{*'\})$ (again we use subscripts when needed for avoiding confusion between the moves of A and $\{*, *'\}$). The PBG A *with* B is $A \,\&\, B = (-, A^- \cup B^-, A^+ \cup B^+, S_A \cup S_B)$. The PBG A *par* B is $A \parr B = (-, A^- \cup B^- \cup (A^- \times B^-), A^+ \cup B^+ \cup (A^+ \times B^+), S_A \otimes_{A,B} S_B)$. The interpretation of positives is defined using duality. Linear implication is defined as usual by setting $A \multimap B = A^\perp \parr B$ (according to logical polarities, A must be positive and B negative).

We also introduce the following non logical constructions. The *negative tensor* of A and B, $A \odot B$ is the PBG $(-, A^- \cup B^-, A^+ \cup B^+, S_A \odot S_B)$. The *negative linear map* from A to B, $A \multimap\!\!\!\!\triangleright B$ is the PBG $(-, A^+ \cup B^-, A^- \cup B^+, S)$ where $S = \{w \in ((A^+ \cup B^-) \cdot (A^- \cup B^+))^*, w \upharpoonright A \in S_A \text{ and } w \upharpoonright B \in S_B\}$. This S is tree-isomorphic to the border of $\downarrow A \multimap B = \uparrow(A^\perp) \parr B$. The *negative tensor unit*, $\int$, is the PBG $(-, \emptyset, \emptyset, \{\varepsilon\})$.

There are many (tree) isomorphisms between the borders associated to these constructions. (Polarized tree isomorphisms are in fact isomorphisms of games in the category to be defined later). Some of them express standard associativity, commutativity, neutrality and distributivity properties.

The other important isomorphisms are[1]:

$$\downarrow \top = 0 \tag{3}$$

$$\downarrow(A \odot B) \cong \downarrow A \otimes \downarrow B \quad (\text{and } \downarrow\!\int = 1) \tag{4}$$

$$A \multimap\!\!\!\!\triangleright \perp = \uparrow(A^\perp) \tag{5}$$

$$\int \multimap\!\!\!\!\triangleright A = A \tag{6}$$

$$A \parr \int \cong A \parr \top \cong \top \tag{7}$$

$$\text{If } \varepsilon \notin S_A \text{ and } \varepsilon \notin S_B \text{ then } A \multimap\!\!\!\!\triangleright B \cong (\downarrow A)^\perp \parr B = (\downarrow A) \multimap B. \tag{8}$$

Definition 3 (linear/affine, full, terminated). *A PBG A is* linear *(resp.* affine*) if $\varepsilon \notin S_A$ (resp. $\varepsilon \in S_A$). A PBG is* full *if for each play p in P_A there exist $s, s' \in S$ such that $s \wedge s' = p$. A PBG is* terminated *if no terminated play is the prefix of another terminated play.*

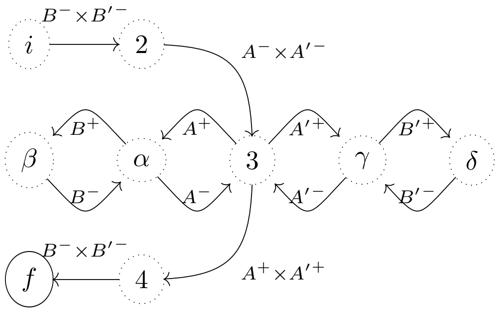
[1] In [15], $\int = \top$ and the same isomorphisms hold.

The interpretation of MALLpol proofs is inductively defined by cases on the last rule, following the pattern given in [15]. As in polarized games we use central strategies to interpret sequents containing one positive formula[2].

Definition 4 (central strategy). *A strategy x in $A \multimap B$ is* central *if, for each element s of x, the first positive move of s is in A^- and the last negative move of s is in A^+.*

A category NG of negative PBGs is defined. The morphisms from A to B are just the strategies of $A \multimap B$. Identity morphisms are, as usual, copycat strategies. In NG, a copycat strategy is central. Every isomorphism is a copycat and defines a unique tree isomorphism between the borders. Conversely, a tree isomorphism between borders of PBGs defines a unique isomorphism, in NG.

The composition $x \fatsemi y$ of two strategies, $x \subseteq S_{A \multimap B}$ and $y \subseteq S_{B \multimap C}$, is defined *pointwise*: $x \fatsemi y = \{s \fatsemi s' \mid s \in x, s' \in y, s{\upharpoonright}B = s'{\upharpoonright}B\}$, where $s \fatsemi s'$ is the projection on $A \cup C$ of the *unique* word t in $(A \cup B \cup C)^*$ such that $t{\upharpoonright}A \cup B = s$ and $t{\upharpoonright}B \cup C = s'$. The word t is called a *witness* of the composition of x and y. If $s'' \in x \fatsemi y$ then there is a unique witness t such that $t{\upharpoonright}A \cup C = s''$. Remark that one has a similar property in coherence spaces or in hypercoherences.



An important point about composition is that it cannot be defined by, first, taking the usual game composition of the even prefix closure of x and y, and then, restrict the result to the border of $A \multimap C$. For non terminated PBGs, this would in general lead to a non associative operation.

We also define a sub-category CNG of NG where objects are linear negative terminated PBGs and morphisms are central strategies.

The *par* of two central strategies, $x \subseteq S_{A \multimap B}$ and $y \subseteq S_{A' \multimap B'}$, is a central strategy $x \parr y$, equal to the set of words w accepted by the automaton above and such that $w{\upharpoonright}(A \multimap B) \in x$ and $w{\upharpoonright}(A' \multimap B') \in y$.

Proposition 5. *The operation $\parr$ is a bifunctor of CNG and this category is symmetric monoidal for the structure $(\parr, \bot)$. The categories NG and CNG are also Cartesian for the with and have $\top$ as terminal object.*

We interpret proofs of $\vdash N_1, \ldots, N_n$ as strategies of $N_1 \parr \ldots \parr N_n$ and proofs of $\vdash P, N_1, \ldots, N_n$ as central strategies of $P^\perp \multimap (N_1 \parr \ldots \parr N_n)$.

The interpretation of additive and *par* rules are as usual. Axioms are interpreted as identity morphisms and cuts are interpreted as composition. If x and y are the central strategies interpreting, respectively, a proof π of $\vdash P, N$ and a proof π' of $\vdash P', N'$, then the application of a tensor rule between π and π' is interpreted as $x \parr y$. The negative shift rule (*i.e.* dereliction)

[2] This terminology has been introduced by Laurent in [15] and refers to Selinger's control categories [19].

is interpreted as a composition with the isomorphism $P^\perp \multimap N \cong \uparrow P \parr N$. If x is the interpretation of a proof π of $\vdash \uparrow P_1, \ldots, \uparrow P_k, N$ then π followed by a positive shift rule (*i.e.* promotion) is interpreted as the central strategy $\downarrow x \subseteq (\downarrow N)^\perp \multimap (\uparrow P_1 \parr \ldots \parr \uparrow P_k)$, equal to:

$$\{(*_1, \ldots, *_k) * n \cdot m \cdot n' *' (*'_1, \ldots, *'_k) \mid (n, *_1, \ldots, *_k) \cdot m \cdot (n', *'_1, \ldots, *'_k) \in x\}. \quad (9)$$

2.1 Reversibility

All the constructions involved in the PBG semantics of MALLpol present a symmetry between the beginning and the end of terminated plays. This symmetry can be exploited to show a reversibility property of the semantics. We do not know if this reversibility property has a deep meaning.

Let $\frown$ be an operation reversing letters in words ($\overset{\frown}{\varepsilon} = \varepsilon$ and $(s \cdot a)^{\frown} = a \cdot \overset{\frown}{s}$).

The *reverse* game of a PBG A is the PBG $\overset{\frown}{A} = (\overline{\epsilon_A}, A^-, A^+, (S_A)^{\frown})$.

The reverse of a strategy in A has no reason to be a strategy in $\overset{\frown}{A}$ or in $\overset{\frown}{A}^\perp$.

Proposition 6 (Reversibility). *If A is the PBG interpreting a MALLpol formula F without atoms then $A^\perp \cong \overset{\frown}{A}$. Let ϕ be the associated tree isomorphism from $S_{A^\perp} = S_A$ to $S_{\overset{\frown}{A}}$ (it just consists in exchanging the elements $*$ and $*'$ in each letter of each word). If x is the interpretation of a proof π of F in MALLpol then the two sets $\overset{\frown}{x}$ and $\phi(x)$ are equal. Provided atoms enjoy the same property (on formulæ), this extends to MALLpol with atoms.*

2.2 Projection on hypercoherences

We adapt the hypercoherence semantics of MALL to MALLpol as follows. If X is a hypercoherence then $\downarrow X$ (resp. $\uparrow X$) is the same hypercoherence, but where each element a of the web of X is renamed as the singleton multiset $[a]$. Hence $|\downarrow X|$ (resp. $|\uparrow X|$) equals $\{[a] \mid a \in |X|\}$. Up to this renaming of elements of the webs, shift rules leave unchanged interpretation of proofs in hypercoherences.

We define a projection of terminated plays in the interpretation of a MALLpol formula on hypercoherences, inductively as follows. For a formula A, p_A denotes the projection. We set $p_{A^\perp}(s) = p_A(s)$, $p_\perp(**') = *$, $p_\alpha(s \cdot a) = a$ and: if $A = B_1 \,\&\, B_2$ and $s \in S_{B_i}$ then $p_A(s) = p_{B_i}(s)$; if $A = B \parr C$ then $p_A(s) = (p_B(s \upharpoonright B), p_C(s \upharpoonright C))$; if $A = \uparrow B$ then $p_A(* \cdot s \cdot *') = [p_B(s \upharpoonright B)]$.

Proposition 7. *Let F be a formula of MALLpol without atoms and π be a proof of F. Then p_F maps the PBG interpretation of π (resp. F) to the hypercoherence interpretation of π (resp.to the web of F). Provided atoms enjoy the same property (on formulæ), this extends to MALLpol with atoms.*

The projection of a strategy in the PBG interpretation of a formula is not, in general, a clique in the hypercoherence interpretation of this formula. For instance, let F be the formula $\downarrow \perp \otimes \downarrow \uparrow (1 \oplus 1)$. Let $A = \downarrow_4 \uparrow_3 (1_1 \oplus 1_2) \otimes \downarrow_6 \perp_5$

be the PBG interpreting F, where the indices separate the various copies of moves, and X be the hypercoherence interpretation of F. Let $s = (*_4, *_6) \cdot w_1 \cdot m \cdot (*'_4, *'_6)$ and $t = (*_4, *_6) \cdot m \cdot w_2 \cdot (*'_4, *'_6)$, where $m = *_2 *'_2$ and $w_i = *_3 *_i *'_i *'_3$ be two terminated plays in A. Then $x = \{s, t\}$ is a strategy but $p(x) = \{([[*'_1]], [*']), ([[*'_2]], [*'])\}$ is not a clique in X since $*'_1$ and $*'_2$ are strictly incoherent in the hypercoherence $1_1 \oplus 1_2$.

3 PBG semantics of ILL and LLpol

A sub-category of NG, ANG, turns out to be a *new Seely* (see [4]) categorical model of ILL. Objects of ANG are affine negative PBGs, and morphisms are strategies containing the empty word. The tensor is interpreted by the negative tensor product. The terminal object of the category is then the unit of the negative tensor product. So, in this semantics, $\top = 1$. The PBGs interpreting formulæ are full but not terminated. ANG admits a comonad structure where the *"of course"*, $\sharp_1$, stands in-between the one of sequential algorithms and the Abrambsky-Jagadeesan-Malacaria ([1]), AJM, for short) or Hyland-Ong ([11], HO, for short) constructions. We describe its action on objects below.

By commutativity and associativity, the binary operation $\odot$ on words can be generalized to an n-ary operation on words on disjoint alphabets. We write $\bigodot\{p_1, \ldots, p_n\}$ for $p_1 \odot \cdots \odot p_n$. We adopt the convention that if $n = 0$ then the resulting set of words is $\{\varepsilon\}$ (which is neutral for $\odot$). We also generalize this operation in the case where alphabets are not disjoint. We set

$$\bigodot_A [p_1, \ldots, p_n] = \left(\bigodot_{A_1, \ldots, A_n} \{p_1 \times \{1\}, \ldots, p_n \times \{n\}\} \right) \restriction A \qquad (10)$$

where $[p_1, \ldots, p_n] \in \mathcal{M}_{\mathrm{fin}}((A \cdot A)^*)$, $A_i = A \times \{i\}$ and where $w \times \{i\}$ is defined inductively by setting $(w' \cdot a) \times \{i\} = (w' \times \{i\}) \cdot (a, i)$ and $\varepsilon \times \{i\} = \varepsilon$. Projecting on A removes these indices. This generalization is well defined, since it does not depend on the enumeration of the elements of the multiset.

We define an embedding operation plg_A, from A^* to $(A^*)^*$, by $\mathrm{plg}_A(\varepsilon) = \varepsilon$ and $\mathrm{plg}_A(m' \cdot a) = \mathrm{plg}_A(m') \cdot (m' \cdot a)$. This operation preserve the prefix ordering.

The PBG $\sharp_1 A$ is equal to $(-, P^-_A, P^+_A, S)$ where:

$$S = \bigcup \left\{ \bigodot_A [\mathrm{plg}_A(p_i) \mid i \in I] \mid I \text{ finite and } \{p_i \mid i \in I\} \text{ strategy in } A \right\}. \qquad (11)$$

Observe that the following two plays in a AJM (resp. HO) *"of course"* game:

$$p = (a_1, 1)(a_2, 1)(a_1, 2)(a_2, 2)(a_3, 1) \quad (\text{resp. } a_1\, a_2\;\; a_1\, a_2\;\; a_3) \qquad (12)$$
$$q = (a_1, 1)(a_2, 1)(a_1, 2)(a_2, 2)(a_3, 2) \quad (\text{resp. } a_1\, a_2\;\; a_1\, a_2\;\; a_3) \qquad (13)$$

correspond to a unique play in the $\sharp_1$ construction:

$$p = q = (a_1)(a_1 a_2)(a_1)(a_1 a_2)(a_1 a_2 a_3). \qquad (14)$$

An operation eff erasing repetitions in words is defined by $\mathrm{eff}(s \cdot a) = \mathrm{eff}(s) \cdot a$ if $a \notin s$, $\mathrm{eff}(s \cdot a) = \mathrm{eff}(s)$ if $a \in s$ and $\mathrm{eff}(\varepsilon) = \varepsilon$.

The construction corresponding to the sequential algorithms' *"of course"* in bordered games would have been to take $\mathrm{eff}(S_{\sharp_1})$ instead of $S_{\sharp_1}$. We denote this construction by $\sharp_a$. Strangely enough $\sharp_a$ is not functorial for non terminated PBGs : in ILL the composition $\sharp_a f \fatsemi \sharp_a g$ may produce plays which are not terminated plays in $\sharp_a(f \fatsemi g)$.

Proposition 8 below shows that the PBG semantics of MALLpol is also a semantics for LLpol (just take the comonad structure associated with $\sharp_1$).

Proposition 8. *If ANG is a new Seely category for a comonad* $(\sharp, \mathrm{dig}, \mathrm{der})$*, then our PBG model of MALLpol extends into a model of LLpol where* $!A = \downarrow \sharp A$.

The arrow of simple types $A \to B$ is interpreted by $(\downarrow \sharp A) \multimap B$ in the LLpol model and by $\sharp A \multimap B$ in the ILL model (these two games are isomorphic).

For the purpose of extending the PBG model of MALLpol to LLpol, one can also use the $\sharp_a$ construction. In fact, we do not really need $\sharp_a$ to be functorial for all objects of ANG but only for negative terminated PBGs as an hypothesis when proving the last proposition.

For $\sharp_1$, as for $\sharp_a$, the reversibility result (Proposition 6) and projection result (Proposition 7) do not extend to LLpol.

The natural way of extending the projection to exponentials would be to associate to a play of $!N$ the projection of its underlying finite strategy in N which is a finite set of points of N.

There are two reasons for not being able to extend Proposition 7 to exponentials. First, with hypercoherences as target, the projection is not defined for all terminated plays. Second, if using the relational model as target, then, with $\sharp_1$ or $\sharp_a$ exponentials at the source, there are points in interpretation of formulæ and proofs which are not in the image of the projection. This is due to the fact that the hypercoherence model and our PBG models are *uniform*: they refer to the notion of agent in the interpretation of exponential formulæ.

For hypercoherences, the points of a $!N$ formula are the finite cliques of N and the projection of a strategy has no reason to be a clique. Hence there are plays in the PBG interpretation of $!N$ to which we cannot naturally associate a point in the hypercoherence $!N$. This prevents the projection from being defined on exponentials when using (standard) hypercoherences as target.

In fact, extending the projection to LLpol was our main motivation in introducing non uniform hypercoherences (see [5]).

With the $\sharp_1$ or $\sharp_a$ based exponentials interpretations in PBGs and with non uniform hypercoherences as target, the projection is well-defined. But the exponentials $\sharp_1$ and $\sharp_a$ require that a play in $!N$ is built from a finite strategy in N. Hence, there are points in $!N$ which are not the image of a play of $!N$ by the projection and such points can occur in the interpretation of proofs. For instance, the point $a = ([([*'], []), ([], [*'])])$ of $P = !(?1 \parr ?1)$ has no counterpart in the PBG interpretation of this formula. As a consequence, the projection of the PBG interpretation of the identity proof of $\vdash P, P^{\perp}$ misses the point $([a], a)$ of its non uniform hypercoherence interpretation.

It is possible to use a non uniform Hyland-Ong style exponential for PBGs in which the reversibility and full projection properties extends to LLpol. This is a work in progress which supposes the introduction of pointers in PBGs such that, in a play, a move can point to several previous moves.

4 Hypercoherences game structures

Infinite coherence. Let X be a hypercoherence. An *infinitely coherent* (resp. infinitely incoherent) subset of X is a non empty directed union on Γ (resp. $\Gamma^{\perp}$) and this subset is strictly infinitely coherent if non reduced to a singleton. Observe that a coherent (resp. incoherent) subset of X is infinitely coherent (resp. infinitely incoherent). In absence of second order, all hypercoherences we use when doing semantics satisfy a convenient property of *local finiteness* we do not recall (see [9]). If X is locally finite then the set of strictly infinitely coherent subets of X, $\Delta^*(X)$, and the set of infinitely incoherent subsets of X, $\Delta(X)$ form a partition of the set of subsets of $|X|$ of cardinality greater than 2. From now on, hypercoherences are always supposed to be locally finite.

Towers. We define a binary relation, $\vdash_X$, on $\mathcal{P}^*(|X|)$ by setting $x \vdash_X y$ when $y \subsetneq x$ and either x is strictly infinitely incoherent in X and y is a maximal infinitely coherent subset of x or x is strictly infinitely coherent in X and y is a maximal infinitely incoherent subset of x.

A *move* on X is a vertex y of the directed acyclic graph $(\mathcal{P}^*(X), \vdash_X)$ such that there exists a directed path $|X| = x_1 \vdash_X x_2 \ldots \vdash_X x_n = y$ from $|X|$ to y in this graph. Local finiteness of X implies that every oriented path in $(\mathcal{P}^*(|X|), \vdash_X)$ is finite. A *tower* on X is a (finite) directed path from $|X|$ to a singleton. A hypercoherence X is *serial parallel* if for each $a \in |X|$ there exists a unique tower on X ending on $\{a\}$.

The *tower graph* on X, denoted by $G(X)$, is the complete subgraph of $(\mathcal{P}^*(X), \vdash_X)$ whose set of vertices is the set $M(X)$ of moves on X. In a tower, moves alternate between $M(X) \cap \Delta^*$, the positive moves, and $M(X) \cap \Delta^{*\perp}$, called negative moves, except the last move, a singleton, which we equip with a relative polarity in the tower with respect to the alternation. The set of towers on X, denoted by $T(X)$, defines a tree, the *tower tree* of X.

A *negative hypercoherence* is a hypercoherence whose web contains at least two points and such that each tower ends on a positive move (*i.e.* singletons are always positive). *Positive* hypercoherences are defined dualy.

Definition 9. *If X is a polarized hypercoherence its associated PBG,* PBG(X), *is $(\epsilon_X, M^-(X), M^+(X), S(X))$ where ϵ_X is the polarity of X, $M^-(X)$ and $M^+(X)$ are respectively the set of negative and positive moves on X and $S(X)$ is the* game structure *of X: either $T(X)$ if $|X| \in M^{\epsilon_X}(X)$ or the forest obtained by erasing the root $|X|$ of $T(X)$, otherwise.*

Proposition 10. *If X and Y are disjoints polarized hypercoherences then:*

1. $\mathrm{PBG}(X^{\perp}) = \mathrm{PBG}(X)^{\perp}$,
2. *if* $X < 0$ *and* $Y < 0$ *then* $\mathrm{PBG}(X \mathbin{\&} Y) = \mathrm{PBG}(X) \mathbin{\&} \mathrm{PBG}(Y)$,
3. *if* $X > 0$ *and* $Y > 0$ *then* $\mathrm{PBG}(X \otimes Y) \cong \mathrm{PBG}(X) \otimes \mathrm{PBG}(Y)$,
4. *if* $X < 0$ *and* $Y < 0$ *then* $\mathrm{PBG}(X \otimes Y) \cong \mathrm{PBG}(X) \odot \mathrm{PBG}(Y)$ *et*
5. *if* $X < 0$ *and* $Y < 0$ *then* $\mathrm{PBG}(X \multimap Y) \cong \mathrm{PBG}(X) \twoheadrightarrow \mathrm{PBG}(Y)$.

By duality, this last case amounts to say that if $X < 0$ and $Y > 0$ then $\mathrm{PBG}(X \otimes Y) \cong {\downarrow} \mathrm{PBG}(X) \otimes \mathrm{PBG}(Y)$.

There is no construction in hypercoherences which correspond to the interpretation of polarity shifts $\downarrow$ and $\uparrow$ in PBGs through an equality like $\mathrm{PBG}({\downarrow} X) = {\downarrow} \mathrm{PBG}(X)$ because the PBGs built from hypercoherences are full and terminated but the polarity shift of a full and terminated non empty PBG is never full.

For the exponentials results are limited. First, to be able to describe simply the towers on $!X$ by means of operations on the set of words $T(X)$, we need to assume that X is serial parallel and this property is not preserved by logical connectives. Second $!X$ is not, in general, a polarized hypercoherence (even if X is negative and serial parallel). But, if we only consider exponentials inside intuitionistic implications (given by the equality $A \to B = !A \multimap B$) then this second limitation is circumvented.

Proposition 11. *If X and Y are two disjoint negative hypercoherences, and if X is serial parallel then* $\mathrm{PBG}(!X \multimap Y) \cong ({\downarrow} \sharp_a \mathrm{PBG}(X)) \multimap \mathrm{PBG}(Y)$.

5 Conclusion

In this paper, we present the polarized bordered game model together with a projection of games onto hypercoherences. This projection commutes to the interpretation of proofs in MALLpol. Extending this commutation result to exponentials requires non uniform game and hypercoherence models.

Our projection relates the sets of plays of games with the webs of hypercoherences, thanks to the introduction of a set of terminated plays. The coherence structures over these webs are still to be related with the dynamical structure games. Our work on tower unfolding is a first attempt in that direction.

In [17], Melliès presents games as directed acyclic graphs from which he extracts hypercoherences. On simple types, the hypercoherence extracted from a game interpreting a type is the hypercoherence interpreting this type. His theory involves a partial projection from plays to points in hypercoherences which associates a web to each game, and an operation on graphs which allows to define coherences on these webs and relates strategies on graph games with cliques in associated hypercoherences. Our intuition about this last operation is that it is a reverse for the operation which, to each hypercoherence X, associates its tower graph $G(X)$ but forgets everything about moves except their polarities. We think that polarized bordered games and hypercoherences unfolding might help in extending Melliès' results to LLpol.

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